ON SOME SPACES USED IN CONVEX AND INTERVAL ANALYSIS

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Abstract: Algebraic systems abstracting properties of convex bodies and intervals, with respect to addition and multiplication by scalars, known as quasilinear spaces, are studied axiomatically. We show that every quasilinear space with group structure is a direct sum of a vector space and a symmetric quasivector space; a complete characterization of the latter spaces in the finite dimensional case is given.

Key words: convex bodies, intervals, vector space, quasilinear space, quasivector space, symmetric quasivector space.

1 Introduction

The set-theoretic operations for addition (also called vector or Minkowski addition) and multiplication by scalars play important roles in convex and interval analysis. The abstract study of these operations leads to the concept of quasilinear space, cf. [2]–[6], [8]–[11]. A quasilinear space over the field of reals can be defined as an additive abelian monoid with cancellation law endowed with multiplication by scalars obeying the four relations of linear spaces with a modified second distributive law (the distributive relation is required to hold only for nonnegative scalars). Quasilinear spaces can be embedded in (additive) groups; thereby an isomorphic extension of the multiplication by scalars leads to a special type of quasilinear spaces (those with group structure), here called quasivector spaces.

It has been shown that every quasivector space is a direct sum of a vector space and a symmetric quasivector space. We demonstrate that symmetric quasivector spaces are close to vector spaces, which enables us to transfer basic concepts of vector spaces to symmetric quasivector spaces. Here we study only the arithmetic operations addition and multiplication by scalars; the important inclusion relation has not been considered.

2 Quasivector Spaces

Quasilinear Spaces. By $\mathbb{R}$ we denote the set of reals; we use the same notation for the linearly ordered (l. o.) field of reals $\mathbb{R} = (\mathbb{R}, +, \cdot, \leq)$. Throughout the paper the real field of scalars $\mathbb{R}$ can be replaced by any other linearly ordered field. For any integer $n \geq 1$ we denote by $\mathbb{R}^n$ the set of all n-tuples $(\alpha_1, \alpha_2, ..., \alpha_n)$, where $\alpha_i \in \mathbb{R}$; $(\alpha_1, \alpha_2, ..., \alpha_n) = (\beta_1, \beta_2, ..., \beta_n)$ means $\alpha_1 = \beta_1, \alpha_2 = \beta_2, ..., \alpha_n = \beta_n$. The set $\mathbb{R}^n$ forms a vector space $\mathbb{V}^n = (\mathbb{R}^n, +, \mathbb{R}, \cdot)$ under the operations of addition and multiplication by scalars:

$$\begin{aligned}(\alpha_1, \alpha_2, ..., \alpha_n) + (\beta_1, \beta_2, ..., \beta_n) &= (\alpha_1 + \beta_1, \alpha_2 + \beta_2, ..., \alpha_n + \beta_n), \\ \gamma \cdot (\alpha_1, \alpha_2, ..., \alpha_n) &= (\gamma\alpha_1, \gamma\alpha_2, ..., \gamma\alpha_n), \ \gamma \in \mathbb{R}.\end{aligned}$$

Definition 1. *Let $(\mathcal{M}, +)$ be an abelian monoid with cancellation law. Assume that a mapping (multiplication by scalars) "$*$" is defined on $\mathbb{R} \times \mathcal{M}$ satisfying: i) $\gamma * (A + B) = \gamma * A + \gamma * B$, ii) $\alpha * (\beta * C) = (\alpha\beta) * C$, iii) $1 * A = A$, iv) $(\alpha + \beta) * C = \alpha * C + \beta * C$, if $\alpha\beta \geq 0$. The algebraic system $(\mathcal{M}, +, \mathbb{R}, *)$ is called a* (cancellative) quasilinear space over $\mathbb{R}$.

Since $(\mathcal{M}, +)$ is not assumed to be a group, there is no opposite in general, that is, for some $A \in \mathcal{M}$ the equation $A + X = 0$ may not have a solution X. The operator $\neg A = (-1) * A$ is called *negation*. We write $A \neg B = A + (\neg B)$; note that $A \neg A = 0$ may not generally hold.

An element $A \in \mathcal{M}$, such that $A \neg A = 0$, is called *linear* or *distributive*; in such case $\mathrm{opp}(A) = \neg A$. We denote $\mathcal{M}' = \{A \in \mathcal{M} \mid A \neg A = 0\}$. An element $A \in \mathcal{M}$, such that $\neg A = A$, is called *(centrally) symmetric*; we denote $\mathcal{M}'' = \{A \in \mathcal{M} \mid \neg A = A\}$.

Using the group extension method every quasilinear space $(\mathcal{M}, +, \mathbb{R}, *)$ can be embedded into an abelian difference (quotient) group $(\mathcal{D}(\mathcal{M}), +)$, where $\mathcal{D}(\mathcal{M}) = \mathcal{M}^2/\!\sim$ is the *difference (quotient) set* of $\mathcal{M}$ consisting of all pairs (A, B) factorized by the congruence relation $\sim$: $(A, B) \sim (C, D)$ iff $A + D = B + C$, for all $A, B, C, D \in \mathcal{M}$. Addition in $\mathcal{D}(\mathcal{M})$ is defined by $(A, B) + (C, D) = (A + C, B + D)$. The neutral (null) element of $\mathcal{D}(\mathcal{M})$ is the class (Z, Z), $Z \in \mathcal{M}$; due to the existence of null element in $\mathcal{M}$, we have $(Z, Z) \sim (0, 0)$. The opposite element to $(A, B) \in \mathcal{D}(\mathcal{M})$ is $\mathrm{opp}(A, B) = (B, A)$. The mapping $\varphi : \mathcal{M} \longrightarrow \mathcal{D}(\mathcal{M})$ defined for $A \in \mathcal{M}$ by $\varphi(A) = (A, 0) \in \mathcal{D}(\mathcal{M})$ is an *embedding* of monoids. We *embed* $\mathcal{M}$ in $\mathcal{D}(\mathcal{M})$ by identifying $A \in \mathcal{M}$ with the equivalence class $(A, 0) \sim (A + X, X)$, $X \in \mathcal{M}$; all elements of $\mathcal{D}(\mathcal{M})$ admitting the form $(A, 0)$ are called *proper*. The set of all proper elements of $\mathcal{D}(\mathcal{M})$ is $\varphi(\mathcal{M}) = \{(A, 0) \mid A \in \mathcal{M}\} \cong \mathcal{M}$.

Multiplication by scalars "$*$" is extended from $\mathbb{R} \times \mathcal{M}$ to $\mathbb{R} \times \mathcal{D}(\mathcal{M})$ by means of the following natural definition of $* : \mathbb{R} \times \mathcal{D}(\mathcal{M}) \longrightarrow \mathcal{D}(\mathcal{M})$:

$$\gamma * (A, B) = (\gamma * A, \gamma * B), \quad A,\ B \in \mathcal{M}, \quad \gamma \in \mathbb{R}. \tag{1}$$

In particular, multiplication by the scalar -1 in $\mathcal{D}(\mathcal{M})$, called *negation*, is denoted by $\neg(A, B) = (-1) * (A, B) = (\neg A, \neg B), \quad A,\ B \in \mathcal{M}$.

In the sequel we shall use lower case roman letters to denote the elements of quasilinear spaces of group structure, such as $\mathcal{D}(\mathcal{M})$, writing e. g. $a = (C, D), C, D \in \mathcal{M}$.

2.1 Quasivector Spaces: Axiomatic Definition

Definition 2. *A* quasivector space (over the l. o. field $\mathbb{R}$)*, denoted* $(\mathcal{Q}, +, \mathbb{R}, *)$*, is an abelian group* $(\mathcal{Q}, +)$ *with a mapping (multiplication by scalars) "$*$": $\mathbb{R} \times \mathcal{Q} \longrightarrow \mathcal{Q}$, such that for $a, b, c \in \mathcal{Q}, \quad \alpha, \beta, \gamma \in \mathbb{R}$:*

$$\gamma * (a + b) = \gamma * a + \gamma * b, \tag{2}$$
$$\alpha * (\beta * c) = (\alpha\beta) * c, \tag{3}$$
$$1 * a = a, \tag{4}$$
$$(\alpha + \beta) * c = \alpha * c + \beta * c, \quad \textit{if} \ \ \alpha\beta \geq 0. \tag{5}$$

Proposition 1. *[4] Let $(\mathcal{M}, +, \mathbb{R}, *)$ be a quasilinear space over $\mathbb{R}$, and let $(\mathcal{Q}, +)$, $\mathcal{Q} = \mathcal{D}(\mathcal{M})$, be the induced abelian group. Let $* : \mathbb{R} \times \mathcal{Q} \longrightarrow \mathcal{Q}$ be multiplication by scalars defined by (1). Then $(\mathcal{Q}, +, \mathbb{R}, *)$ is a quasivector space over $\mathbb{R}$.*

As before we denote $\neg a = (-1) * a$. From $\mathrm{opp}(a) + a = 0$ we obtain $\neg\mathrm{opp}(a) \neg\ a = 0$, that is $\neg\mathrm{opp}(a) = \mathrm{opp}(\neg a)$. The element $\neg\mathrm{opp}(a) = \mathrm{opp}(\neg a)$ will be further denoted by a_- and called *dual* or *conjugate* to a, and the corresponding operator will be called *dualization* or *conjugation*.

Relations $\neg\mathrm{opp}(a) = \mathrm{opp}(\neg a) = a_-$ imply $\mathrm{opp}(a) = \neg(a_-) = (\neg a)_-$, which will be shortly denoted by $\mathrm{opp}(a) = \neg a_-$. The last notation will be used to denote symbolically the opposite elements instead of the confusing notation $-a$ meaning opposite in algebra and negation in convex and interval analysis.

A *subspace of a quasivector space* $(\mathcal{Q}, +, \mathbb{R}, *)$ is a quasivector space $(\mathcal{P}, +, \mathbb{R}, *)$, such that $\mathcal{P} \subseteq \mathcal{Q}$. If $(\mathcal{P}, +, \mathbb{R}, *)$ is a subspace of the quasivector space $(\mathcal{Q}, +, \mathbb{R}, *)$ then, of course, $(\mathcal{P}, +)$ is an abelian subgroup of the abelian group $(\mathcal{Q}, +)$. Sum and direct sum of quasivector spaces are defined as in vector spaces. Namely, for two quasivector spaces U, V there is a least subspace containing both U and V, called their sum and written $U + V$. We have $U + V = \{u + v \mid u \in U, v \in V\}$. Let Z be a quasivector space and U, V be subspaces of Z. We say that Z is the *direct sum* of U and V and write $Z = U \oplus V$, if each $z \in Z$ can be uniquely presented in the form $z = u + v$, where $u \in U$, $v \in V$. One can show: 1) a sum $U + V$ is direct, if $u_1 + v_1 = u_2 + v_2$, $u_1, u_2 \in U$, $v_1, v_2 \in V$ imply $u_1 = u_2$, $v_1 = v_2$ (or, equivalently, $u + v = 0$, $u \in U$, $v \in V$ imply $u = 0$, $v = 0$); 2) $Z = U \oplus V \iff Z = U + V$ and $U \cap V = 0$. The elements of $U \oplus V$ are denoted (u, v) or $(u; v)$. Addition in $U \oplus V$ is $(u_1; v_1) + (u_2; v_2) = (u_1 + u_2; v_1 + v_2)$ and multiplication by scalars is $\gamma * (u; v) = (\gamma * u; \gamma * v)$. We have [4]: $\mathcal{H}$ is a subspace of the quasivector space $\mathcal{G}$ if and only if $\mathcal{H} \subset \mathcal{G}$ and $\mathcal{H}$ is closed under "+", "$*$", "$_-$", i. e.: i) $a + b \in \mathcal{H}$ for all $a, b \in \mathcal{H}$; ii) $\alpha * c \in \mathcal{H}$ for all $\alpha \in \mathbb{R}$ and $c \in \mathcal{H}$; iii) $a_- \in \mathcal{H}$ for all $a \in \mathcal{H}$.

2.2 Examples of Quasivector Spaces

Example 1. The system $(\mathcal{K}, +)$ of all convex bodies [12] of a real m-dimensional Euclidean vector space $\mathbb{E}^m$ with set-theoretic (vector, Minkowski) addition: $A + B = \{\alpha + \beta \mid \alpha \in A, \beta \in B\}, A, B \in \mathcal{K}$, is a proper abelian monoid with cancellation law having as a neutral element the origin "0" of $\mathbb{E}^m$. The system $(\mathcal{K}, +, \mathbb{R}, *)$, where "$*$" is the set-theoretic multiplication by real scalars: $\gamma * A = \{\gamma\alpha \mid \alpha \in A\}$, is a quasilinear space (of monoid structure). The monoid $(\mathcal{K}, +)$ induces a group of generalized convex bodies $(\mathcal{D}(\mathcal{K}), +)$, which has been considered by a number of authors, cf. [1], [7], [8], [9]. In [9] and some of the above cited literature the following multiplication by scalars has been used in $(\mathcal{D}(\mathcal{K}), +)$:

$$\gamma \cdot (A, B) = \begin{cases} (\gamma * A, \gamma * B), & \text{if } \gamma \geq 0, \\ (|\gamma| * B, |\gamma| * A), & \text{if } \gamma < 0. \end{cases} \tag{6}$$

As for $\gamma < 0$ we have $\gamma \cdot (A, 0) = (0, |\gamma| * A)$, which is an improper result, (6) is not an extension of the multiplication by scalars in $\mathcal{K}$ and seems to be of little practical value. In [4] we investigate the space $(\mathcal{D}(\mathcal{K}), +, \mathbb{R}, *)$, where "$*$" is defined by (1). We recall that opposite is $\mathrm{opp}(A, B) = (B, A)$, negation is $\neg(A, B) = (\neg A, \neg B)$, and conjugation is $(A, B)_- = (\neg B, \neg A)$. As special case n-dimensional intervals form a quasilinear space [2]–[6], [10], [11], which induces a quasivector space of generalized (directed) intervals, see, e. g. [5].

Example 2. For any integer $k \geq 1$ the set $\mathbb{R}^k$ of all k-tuples $(\alpha_1, \alpha_2, ..., \alpha_k)$, where $\alpha_i \in \mathbb{R}$ and $(\alpha_1, \alpha_2, ..., \alpha_k)$, $(\beta_1, \beta_2, ..., \beta_k)$ are distinct unless $\alpha_1 = \beta_1, \alpha_2 = \beta_2, ..., \alpha_k = \beta_k$, forms a quasivector space over $\mathbb{R}$ under the following operations

$$(\alpha_1, \alpha_2, ..., \alpha_k) + (\beta_1, \beta_2, ..., \beta_k) = (\alpha_1 + \beta_1, \alpha_2 + \beta_2, ..., \alpha_k + \beta_k), \quad (7)$$
$$\gamma * (\alpha_1, \alpha_2, ..., \alpha_k) = (|\gamma|\alpha_1, |\gamma|\alpha_2, ..., |\gamma|\alpha_k),\ \gamma \in \mathbb{R}. \quad (8)$$

This quasivector space will be denoted by $\mathbb{S}^k = (\mathbb{R}^k, +, \mathbb{R}, *)$ and called the *canonical symmetric quasivector space*. Note that multiplication by -1 (negation) in $\mathbb{S}^k$ is the same as identity while the opposite operator is the same as conjugation:

$$\mathrm{opp}(\alpha_1, \alpha_2, ..., \alpha_k) = (\alpha_1, \alpha_2, ..., \alpha_k)_- = (-\alpha_1, -\alpha_2, ..., -\alpha_k). \quad (9)$$

Denoting $\mathbb{S} = \mathbb{S}^1$, we have $\mathbb{S}^k = \mathbb{S} \oplus \mathbb{S} \oplus \ldots \oplus \mathbb{S}$.

Example 3. Consider the set of infinite sequences $(\alpha_1, \alpha_2, ...)$, $\alpha_i \in \mathbb{R}$, with addition and multiplication by scalars defined as in (7) and (8). We again obtain a quasivector space.

Example 4. The set of all real functions is a quasivector space if we define $f + g$ as the function whose value at x is $f(x) + g(x)$, and $\gamma * f$ as a function whose value at x is

$$\gamma * f(x) = \begin{cases} \gamma \cdot f(x), & \text{if } \gamma \geq 0, \\ |\gamma| \cdot f(-x), & \text{if } \gamma < 0. \end{cases} \quad (10)$$

In particular, negation is: $-1 * f(x) = f(-x)$. Note that in this quasivector space negation is distinct from opposite $\mathrm{opp}(f) = -f$. Note that the composition of opposite and negation $-f(-x)$ is a new operator. The operation (10) appears in the theory of (differences of) support functions, cf. [4], [13]. We note that, if f is the support function of $A \in \mathcal{K}$, then (10) is the support function of the convex body $\gamma * A$; in particular, $-1 * f(x) = f(-x)$ is the support function of $\neg A$.

Example 5. Let $\mathbb{C} = (\mathbb{C}, +, \mathbb{R}, \cdot)$ be the vector space of all complex numbers $c = c_1 + ic_2$ with addition: $(c_1 + ic_2) + (d_1 + id_2) = (c_1 + d_1) + i(c_2 + d_2)$ and multiplication by real scalars: $\gamma \cdot (c_1 + ic_2) = \gamma c_1 + i\gamma c_2$. Opposite is $-c = -c_1 - ic_2$. One introduces in $\mathbb{C}$ conjugate elements by means of: $c_- = \overline{c} = c_1 - ic_2$; in particular $\overline{i} = -i$. Define a new multiplication by scalars in $\mathbb{C}$ by:

$$\gamma * c = \begin{cases} \gamma \cdot c, & \text{if } \gamma \geq 0, \\ \gamma \cdot \overline{c}, & \text{if } \gamma < 0. \end{cases}$$

The system $\mathbb{C}^* = (\mathbb{C}, +, \mathbb{R}, *)$ is a quasivector space. Negation in $\mathbb{C}^*$ is $\neg c = (-1) * c = -\overline{c} = -(c_1 - ic_2) = -c_1 + ic_2$. We have $\mathbb{C}^* = \mathbb{V}^1 \oplus \mathbf{Im}$, where

$\mathbf{Im} = (\mathbf{Im}, +, \mathbb{R}, *)$ is the quasivector space of purely imaginary numbers. Note that in **Im** negation is same as identity, whereas conjugation is same as opposite.

Example 6. Consider the direct sum $\mathbb{V}^l \oplus \mathbb{S}^k$ of the l-dimensional vector space $\mathbb{V}^l = (\mathbb{R}^l, +, \mathbb{R}, \cdot)$ and the quasivector space $\mathbb{S}^k = (\mathbb{R}^k, +, \mathbb{R}, *)$ from Example 2. The elements of $\mathbb{V}^l \oplus \mathbb{S}^k$ are n-tuples, $n = l + k$, of the form $(\lambda_1, ..., \lambda_l; \lambda_{l+1}, ..., \lambda_{l+k})$. Addition and multiplication by scalars ($\gamma \in \mathbb{R}$) are:

$$(\lambda_1, ..., \lambda_l; \lambda_{l+1}, ..., \lambda_{l+k}) + (\mu_1, ..., \mu_l; \mu_{l+1}, ..., \mu_{l+k})$$
$$= (\lambda_1 + \mu_1, ..., \lambda_l + \mu_l; \lambda_{l+1} + \mu_{l+1}, ..., \lambda_{l+k} + \mu_{l+k}),$$
$$\gamma * (\lambda_1, ..., \lambda_l; \lambda_{l+1}, ..., \lambda_{l+k}) = (\gamma\lambda_1, ..., \gamma\lambda_l; |\gamma|\lambda_{l+1}, ..., |\gamma|\lambda_{l+k}).$$

As direct sum of two quasivector spaces, $\mathbb{V}^l \oplus \mathbb{S}^k$ is a quasivector space. Negation

$$(-1) * (\lambda_1, ..., \lambda_l; \lambda_{l+1}, ..., \lambda_{l+k}) = (-\lambda_1, ..., -\lambda_l; \lambda_{l+1}, ..., \lambda_{l+k})$$

is distinct from opposite: $\mathrm{opp}(\lambda_1, ..., \lambda_{l+k}) = (-\lambda_1, ..., -\lambda_l; -\lambda_{l+1}, ..., -\lambda_{l+k})$. The composition of the opposite and negation operators yields: $\mathrm{opp}(\neg(\lambda_1, ..., \lambda_l; \lambda_{l+1}, ..., \lambda_{l+k})) = (\lambda_1, ..., \lambda_l; -\lambda_{l+1}, ..., -\lambda_{l+k})$.

2.3 Rules for Calculation in Quasivector spaces

Let $(\mathcal{Q}, +, \mathbb{R}, *)$ be a quasivector space over $\mathbb{R}$. As $(\mathcal{Q}, +)$ is a group, for every a there exists an opposite element $\mathrm{opp}(a) = \neg a_-$, such that $a \neg a_- = 0$. In a quasivector space the relation $\neg a + a = 0$, may not necessarily hold; indeed, due to the condition $\alpha\beta \geq 0$ in (5) the equality $(-1) * a + 1 * a = (-1 + 1) * a$ may not be true. This means that generally the element $\neg a$ does not coincide with the opposite of a (unlike in a vector space, where negation and opposite coincide). Using the group properties, such as $0 + a = a$, $\mathrm{opp}(a) + a = 0$, $\mathrm{opp}(a + b) = \mathrm{opp}(a) +\mathrm{opp}(b)$, $a + b = a + c \Longrightarrow b = c$, and relations (2)–(5) one can derive rules for calculation in a quasivector space. A list of such rules is summarized in the following

Proposition 2. *Let $(\mathcal{Q}, +, \mathbb{R}, *)$ be a quasivector space over $\mathbb{R}$. For all $\alpha, \beta, \gamma \in \mathbb{R}$ and for all $a, b, c \in \mathcal{Q}$ the following properties hold: 1) $0 * a = 0$; 2) $\gamma * 0 = 0$; 3) $\gamma * \mathrm{opp}(a) = \mathrm{opp}(\gamma * a)$; 4) $\neg(\gamma * a) = (-\gamma) * a$; 5) $\gamma * (a \neg b) = \gamma * a \neg \gamma * b$; 6) $\gamma * a = 0 \Longrightarrow \gamma = 0$ or $a = 0$; 7) $\gamma * a = \gamma * b \Longrightarrow \gamma = 0$ or $a = b$; 8) $(\alpha - \beta) * c = \alpha * c + (-\beta) * c = \alpha * c \neg \beta * c$, $\alpha\beta \leq 0$; 9) $(\sum_{i=1}^n \alpha_i) * c = \sum_{i=1}^n \alpha_i * c$, $\alpha_i \geq 0$, $i = 1, ..., n$; 10) $\alpha * \sum_{i=1}^n c_i = \sum_{i=1}^n \alpha * c_i$.*

The verification of the above properties is trivial. Note that $y = a_-$ is the solution of the equation: $y \neg a = 0$, resp. $\neg y + a = 0$. For $\gamma \in \mathbb{R}$ and

$a, b \in \mathcal{Q}$ we have the following relations using conjugation: $\gamma * (a + b_-) = \gamma * a + \gamma * b_-$; $\gamma * (a \neg b_-) = \gamma * a \neg \gamma * b_-$; $a \neg a_- = \neg a + a_- = 0$; $\gamma * a \neg \gamma * b_- = \gamma * (a \neg b_-) = 0$; $a + b = 0 \Longleftrightarrow a = \neg b_-$; $a + \gamma * b = 0 \Longleftrightarrow a = (-\gamma) * b_- = \neg(\gamma * b_-)$.

We shall make use of the binary set $\Lambda = \{+, -\}$ and the function $\sigma : \mathbb{R} \longrightarrow \Lambda$ defined by:

$$\sigma(\gamma) = \begin{cases} +, & \text{if } \gamma \geq 0, \\ -, & \text{if } \gamma < 0. \end{cases}$$

The "product" $\lambda\mu$, $\lambda, \mu \in \Lambda$, is defined by $++ = -- = +$, $+- = -+ = -$. Denote $a_+ = a$. Then the symbolic notation a_λ for $a \in \mathcal{Q}$, $\lambda \in \Lambda$, makes sense; namely a_λ is either a or a_- according to the binary value of λ. Using this notation one may write rules holding true for all $a, b, c \in \mathcal{Q}, \alpha \in \mathbb{R}, \lambda, \mu, \nu \in \Lambda$, such as: $(a + b)_\lambda = a_\lambda + b_\lambda$; $(a_\mu + b_\nu)_\lambda = a_{\lambda\mu} + b_{\lambda\nu}$; $(\alpha * c_\mu)_\nu = \alpha * c_{\mu\nu}$, e. g., $(\alpha * c_\mu)_\mu = \alpha * c$. The possibility to perform such symbolic transformations justifies the use of the notation a_- for conjugate instead of the traditional notation $\overline{a}$.

Theorem 1. *Let $(\mathcal{Q}, +, \mathbb{R}, *)$ be a quasivector space over $\mathbb{R}$. For $\alpha, \beta \in \mathbb{R}$ and $c \in \mathcal{Q}$ we have:*

$$(\alpha + \beta) * c_{\sigma(\alpha+\beta)} = \alpha * c_{\sigma(\alpha)} + \beta * c_{\sigma(\beta)}. \tag{11}$$

Proof. In the case $\sigma(\alpha) = \sigma(\beta)$ (11) is true by assumption (5). Consider the case $\sigma(\alpha) = -\sigma(\beta)$. Assume that $0 \leq \alpha$, $\beta < 0$ and $0 < -\beta \leq \alpha$. In this subcase we have $0 \leq \alpha + \beta$, so that (11) reads: $(\alpha + \beta) * c = a * c + \beta * c_-$. Using (5), we can write

$$\begin{aligned} \alpha * c + \beta * c_- &= ((\alpha + \beta) - \beta) * c + \beta * c_- \\ &= (\alpha + \beta) * c \neg \beta * c + \beta * c_- \\ &= (\alpha + \beta) * c + \beta * (\neg c + c_-) = (\alpha + \beta) * c, \end{aligned}$$

so that (11) is proved to hold true in this subcase. The remaining subcases are verified similarly. □

2.4 Linear and Symmetric Subspaces

Let $(\mathcal{Q}, +, \mathbb{R}, *)$ be a quasivector space over the l. o. field $\mathbb{R}$. The following theorem is related to similar result by Rådström [9]. Consider the operation "$\cdot$": $\mathbb{R} \times \mathcal{Q} \longrightarrow \mathcal{Q}$ by

$$\alpha \cdot c = \alpha * c_{\sigma(\alpha)},\ \alpha \in \mathbb{R},\ c \in \mathcal{Q}. \tag{12}$$

Theorem 2. *Let $(\mathcal{Q}, +, \mathbb{R}, *)$ be a quasivector space over $\mathbb{R}$. Then $(\mathcal{Q}, +, \mathbb{R}, \cdot)$, with "$\cdot$" defined by (12), is a vector space over $\mathbb{R}$.*

Conversely, in a vector space $(\mathcal{G}, +, \mathbb{R}, \cdot)$ we can define:

$$\alpha * c = |\alpha| \cdot c. \tag{13}$$

Then the induced space $(\mathcal{G}, +, \mathbb{R}, *)$ with "$*$" defined by (13) is a quasivector space, which is not linear in general. From (13) we obtain for $\alpha = -1$ that $(-1) * c = c$, showing that the set $\mathcal{G}$ consists of symmetric elements. We pay a special attention to this important case in the next Section 3.

Recall that an element $a \in \mathcal{Q}$ with $a \neg a = 0$ is called *linear* or *distributive*. An element $a \in \mathcal{Q}$ with $\neg a = a$ is called *symmetric*. It is easy to check that in a quasivector space $\mathcal{Q}$ the subsets of linear and symmetric elements $\mathcal{Q}' = \{a \in \mathcal{Q} \mid a \neg a = 0\}$, resp. $\mathcal{Q}'' = \{a \in \mathcal{Q} \mid a = \neg a\}$ form subspaces of $\mathcal{Q}$.

Definition 3. *Assume that Q is a quasivector space. The space $\mathcal{Q}' = \{a \in \mathcal{Q} \mid a \neg a = 0\}$ is called the* linear (distributive) subspace *of $\mathcal{Q}$ and the space $\mathcal{Q}'' = \{a \in \mathcal{Q} \mid a = \neg a\}$ is called the* symmetric (quasivector) subspace *of $\mathcal{Q}$.*

Below we summarize some of the properties of the linear and symmetric elements:

1. $a \in \mathcal{Q}' \Longleftrightarrow a = a_- \Longleftrightarrow a \neg a = 0 \Longleftrightarrow \neg a = \mathrm{opp}(a)$
$\Longleftrightarrow \exists c \in \mathcal{Q} : a = c + c_-;$
2. $b \in \mathcal{Q}'' \Longleftrightarrow b = \neg b \Longleftrightarrow b + b_- = 0 \Longleftrightarrow b_- = \mathrm{opp}(b)$
$\Longleftrightarrow \exists d \in \mathcal{Q} : b = d \neg d.$

To prove existence, in case 1 take $c = (1/2) * a + s$, where $s \in \mathcal{Q}''$ is arbitrary, and in case 2 take $d = (1/2) * b + t$, where $t \in \mathcal{Q}'$ is arbitrary.

Theorem 3. *For every quasivector space $\mathcal{Q}$ we have $\mathcal{Q} = \mathcal{Q}' \oplus \mathcal{Q}''$. More specifically, for every $x \in \mathcal{Q}$ we have $x = x' + x''$ with unique $x' = (1/2) * (x + x_-) \in \mathcal{Q}'$, and $x'' = (1/2) * (x \neg x) \in \mathcal{Q}''$.*

Proof. Assume $x \in \mathcal{Q}$. Using that $x_- \neg x = 0$ we have $x' + x'' = (1/2) * (x + x_-) + (1/2) * (x \neg x) = (1/2) * (x + x) + (1/2) * (x_- \neg x) = x$. On the other side we have $x' = (1/2) * (x + x_-) \in \mathcal{Q}'$, $x'' = (1/2) * (x \neg x) \in \mathcal{Q}''$. Hence, $\mathcal{Q} = \mathcal{Q}' + \mathcal{Q}''$. Furthermore, $\mathcal{Q}' \cap \mathcal{Q}'' = 0$. Indeed, assume $x \in \mathcal{Q}'$ and $x \in \mathcal{Q}''$. Then we have simultaneously $x \neg x = 0$ and $x = \neg x$, implying $x = 0$. Hence $\mathcal{Q} = \mathcal{Q}' \oplus \mathcal{Q}''$. □

3 Symmetric Quasivector Spaces

3.1 Relation to Vector Spaces

Recall that an element s from a quasivector space, such that $(-1) * s = s$, briefly $\neg s = s$, is called *(centrally) symmetric.* A quasivector space consisting of symmetric elements is called a *symmetric quasivector space.*

A symmetric quasivector space $\mathcal{S}$ can be defined axiomatically as an abelian group with multiplication by scalars (from a l. o. field) satisfying (2)–(5) together with the additional assumption: $\neg a = a$ for all $a \in \mathcal{S}$.

In a symmetric quasivector space we have: $\alpha * c = (-\alpha) * c = |\alpha| * c$. Hence for the induced linear multiplication in a symmetric quasivector space, cf. (12), we can write

$$\alpha \cdot c = |\alpha| * c_{\sigma(\alpha)}. \tag{14}$$

In a vector space one can introduce a quasivector multiplication by scalars via $\alpha * c = |\alpha| \cdot c$, thus using the available operations in a vector space over the l. o. field $\mathbb{R}$.

Substituting $\alpha = -1$ in (13), we obtain $\neg c = 1 \cdot c = c$. This shows that negation coincides with identity (all elements c are symmetric); hence negation can be expressed by means of the operations in the vector space. Also, it is immediately seen that conjugation equals opposite,

$$a_- = \mathrm{opp}(\neg a) = \mathrm{opp}(a). \tag{15}$$

Thus to every vector space over a l. o. field $(\mathcal{G}, +, \mathbb{R}, \cdot)$, we associate the symmetric quasivector space $(\mathcal{G}, +, \mathbb{R}, *)$ with "$*$" defined by (13). In accordance to what was said above we distinguish between the two multiplications by scalars: the *linear multiplication by scalars* "$\cdot$", and the *quasivector* multiplication by scalars "$*$". Note that the two spaces — the original one $(\mathcal{Q}, +, \mathbb{R}, *)$ and the induced one $(\mathcal{Q}, +, \mathbb{R}, \cdot)$ — are generally distinct from each other as they generally have different operations for multiplication by scalars; the quasivector multiplication in a symmetric quasivector space is generally not a linear multiplication.

To summarize: *Every symmetric quasivector space over* $\mathbb{R}$ *generates via (14) a vector space and, vice versa, every linear space over* $\mathbb{R}$ *induces via (13) a symmetric quasivector space.*

We may briefly express the above by saying that both spaces are equivalent. More specifically, the vector space $\mathbb{V}^n = (\mathbb{R}^n, +, \mathbb{R}, \cdot)$ and the symmetric quasivector space $\mathbb{S}^n = (\mathbb{R}^n, +, \mathbb{R}, *)$ are equivalent in the above sense.

3.2 Linear Combinations in Symmetric Quasivector Spaces

Assume that $\mathcal{S} = (\mathcal{S}, +, \mathbb{R}, *)$ is a *symmetric* quasivector space and $(\mathcal{S}, +, \mathbb{R}, \cdot)$ is the associated equivalent vector space. From the vector space $(\mathcal{S}, +, \mathbb{R}, \cdot)$ we may transfer vector space concepts, such as linear combination, linear dependence, basis etc., to the original symmetric quasivector space $(\mathcal{S}, +, \mathbb{R}, *)$. For example, the concept of linear combination obtains the following form.

Let $c^{(1)}, c^{(2)}, ..., c^{(k)}$ be finitely many (not necessarily distinct) elements of $\mathcal{S}$. An element $f \in \mathcal{S}$ of the form

$$f = \alpha_1 * c^{(1)}_{\sigma(\alpha_1)} + \alpha_2 * c^{(2)}_{\sigma(\alpha_2)} + ... + \alpha_k * c^{(k)}_{\sigma(\alpha_k)}, \tag{16}$$

where $\alpha_1, \alpha_2, ..., \alpha_k \in \mathbb{R}$, is called a *linear combination* of $c^{(1)}, c^{(2)}, ..., c^{(k)} \in \mathcal{S}$.

Remarks. 1) Using (12) we see that (16) is a reformulation of the familiar linear combination $f = \sum_{i=1}^{k} \alpha_i \cdot c^{(i)} = \alpha_1 \cdot c^{(1)} + \alpha_2 \cdot c^{(2)} + ... + \alpha_k \cdot c^{(k)}$ from the induced vector space $(\mathcal{S}, +, \mathbb{R}, \cdot)$. 2) It is shown in [4] that the concept of linear combination can be directly extended to an arbitrary (not necessarily symmetric) quasivector space; here we use a more restricted, but simpler approach.

Proposition 3. *Let* $c^{(1)}, c^{(2)}, ..., c^{(k)} \in \mathcal{S}$, $k \geq 1$. *Then the set*

$$\mathcal{H} = \{\sum_{i=1}^{k} \alpha_i * c^{(i)}_{\sigma(\alpha_i)} \mid \alpha_i \in \mathbb{R}\}$$

of all linear combinations of $c^{(1)}, c^{(2)}, ..., c^{(k)}$ *is a subspace of* $\mathcal{S}$.

The proof is elementary — it can be done either by passing to the induced vector space or in terms of quasivector spaces.

The elements $c^{(1)}, c^{(2)}, ..., c^{(k)}$ form a *generating set* for $\mathcal{H}$. We also say that the subspace $\mathcal{H}$ defined in Proposition 3 is *spanned* by $c^{(1)}, c^{(2)}, ..., c^{(k)}$ and write $\mathcal{H} = \text{span}\{c^{(1)}, c^{(2)}, ..., c^{(k)}\}$.

Let $\mathcal{S}$ be a symmetric quasivector space over $\mathbb{R}$. The elements $c^{(1)}, c^{(2)}, ..., c^{(k)} \in \mathcal{S}$, $k \geq 1$, are *linearly dependent (over* $\mathbb{R}$*)*, if there exists a nontrivial linear combination of $\{c^{(i)}\}$, which is equal to 0, i. e. if there exist a system $\{\alpha_i\}_{i=1}^{k}$ with not all α_i equal to zero, such that

$$\alpha_1 * c^{(1)}_{\sigma(\alpha_1)} + \alpha_2 * c^{(1)}_{\sigma(\alpha_2)} + ... + \alpha_k * c^{(k)}_{\sigma(\alpha_k)} = 0. \tag{17}$$

Elements of $\mathcal{S}$, which are not linearly dependent, are *linearly independent*. That is, the elements $c^{(1)}, c^{(2)}, ..., c^{(k)} \in \mathcal{S}$ are *linearly independent*, if (17) is possible only for the trivial linear combination, such that $\alpha_i = 0$ for all $i = 1, ..., k$.

3.3 Linear Mappings in Quasivector Spaces

Let $\mathcal{Q}_1 = (\mathcal{Q}_1, +, \mathbb{R}, *)$, $\mathcal{Q}_2 = (\mathcal{Q}_2, +, \mathbb{R}, *)$ be two quasivector spaces over $\mathbb{R}$ and let $\varphi : \mathcal{Q}_1 \longrightarrow \mathcal{Q}_2$ be a homomorphic (linear) mapping, that is:

$$\begin{aligned}
\varphi(x+y) &= \varphi(x) + \varphi(y), & (18)\\
\varphi(\lambda * x) &= \lambda * \varphi(x), \quad x, y \in \mathcal{Q}_1, \ \lambda \in \mathbb{R}. & (19)
\end{aligned}$$

It is easy to check that $\varphi(x_-) = (\varphi(x))_-$; more generally any linear mapping satisfies:

$$\begin{aligned}
&\varphi(\alpha_1 * x^{(1)}_{\sigma(\alpha_1)} + \alpha_2 * x^{(2)}_{\sigma(\alpha_2)} + ... + \alpha_k * x^{(k)}_{\sigma(\alpha_k)}) = & (20)\\
&\alpha_1 * \varphi(x^{(1)})_{\sigma(\alpha_1)} + \alpha_2 * \varphi(x^{(2)})_{\sigma(\alpha_2)} + ... + \alpha_k * \varphi(x^{(k)})_{\sigma(\alpha_k)},
\end{aligned}$$

where $\alpha_1, \alpha_2, ..., \alpha_k \in \mathbb{R}$, $x^{(1)}, x^{(2)}, ..., x^{(k)} \in \mathcal{Q}_1$. In particular:

$$\varphi(\alpha * x_\lambda + \beta * y_\mu) = \alpha * \varphi(x)_\lambda + \beta * \varphi(y)_\mu, \ x, y \in \mathcal{Q}_1, \ \lambda, \mu \in \mathbb{R}. \qquad (21)$$

Obviously condition (21) completely characterizes a linear mapping and can substitute conditions (18) and (19).

Let $\mathcal{S}$ be a symmetric quasivector space and $x^{(1)}, x^{(2)}, ..., x^{(n)} \in \mathcal{S}$ and let $\mathbb{S}^n = (\mathbb{R}^n, +, \mathbb{R}, *)$ be the canonic symmetric quasilinear space defined in Example 2. It is easy to check that the mapping $\varphi : \mathbb{S}^n \longrightarrow \mathcal{S}$, such that

$$\varphi(\alpha_1, \alpha_2, ..., \alpha_n) = \alpha_1 * x^{(1)}_{\sigma(\alpha_1)} + \alpha_2 * x^{(2)}_{\sigma(\alpha_2)} + ... + \alpha_n * x^{(n)}_{\sigma(\alpha_n)}, \qquad (22)$$

is linear.

Denote $e^{(i)} = (0, 0, ..., 0, 1, 0, ..., 0)$, where the component 1 is on the i-th place. We consider $e^{(i)}$ as elements of $\mathbb{S}^n$, where $\mathrm{opp}(e^{(i)}) = e^{(i)}_-$ and $\neg e^{(i)} = e^{(i)}$. Relation (22) implies

$$\varphi(e^{(i)}) = \alpha_i * x^{(i)}_{\sigma(\alpha_i)}|_{\alpha_i=1} = x^{(i)}, \ i = 1, ..., n. \qquad (23)$$

The mapping φ is the only linear mapping from $\mathbb{S}^n$ to $\mathcal{S}$ with the property (23). Indeed, if (23) holds, then by (20),

$$\begin{aligned}
\varphi(\alpha_1, \alpha_2, ..., \alpha_n) &= \varphi(\sum \alpha_i * e^{(i)}_{\sigma(\alpha_i)})\\
&= \sum \alpha_i * \varphi(e^{(i)})_{\sigma(\alpha_i)} = \sum \alpha_i * x^{(i)}_{\sigma(\alpha_i)}.
\end{aligned}$$

We thus obtain that relation (23): $\varphi(e^{(i)}) = x^{(i)}$, $i = 1, ..., n$, is sufficient to determine the mapping (22). As in the linear case, every mapping of the set $(e^{(1)}, ..., e^{(n)})$ into $\mathcal{S}$ of the form $\varphi(e^{(i)}) = x^{(i)}$, $i = 1, ..., n$, can be extended to a unique linear mapping of $\mathbb{S}^n$ into $\mathcal{S}$.

3.4 Basis in a Symmetric Quasivector Space

Let $\mathcal{S}$ be a symmetric quasivector space over $\mathbb{R}$. The set $\{c^{(i)}\}_{i=1}^{k}$, $c^{(i)} \in \mathcal{S}$, $k \geq 1$, is a *basis* of $\mathcal{S}$, if $c^{(i)}$ are linearly independent and $\mathcal{S} = \mathrm{span}\{c^{(i)}\}_{i=1}^{k}$.

Proposition 4. *Let $\mathcal{S}$ be a symmetric quasivector space over $\mathbb{R}$. A set $\{c^{(i)}\}_{i=1}^{k}$, $c^{(i)} \in \mathcal{S}$, $k \geq 1$, is a basis of $\mathcal{S}$, iff every $f \in \mathcal{S}$ can be presented in the form (16) in a unique way (i. e. with unique scalars α_i).*

Let $\mathcal{S}$ be a symmetric quasivector space over $\mathbb{R}$ and $\{c^{(i)}\}_{i=1}^{k}$ be a basis of $\mathcal{S}$. Assume that $a = \sum_{i=1}^{k} \alpha_i * c_{\sigma(\alpha_i)}^{(i)}$, $b = \sum_{i=1}^{k} \beta_i * c_{\sigma(\beta_i)}^{(i)}$ are two elements of $\mathcal{S}$. Their sum is

$$a + b = \sum_{i=1}^{k} \alpha_i * c_{\sigma(\alpha_i)}^{(i)} + \sum_{i=1}^{k} \beta_i * c_{\sigma(\beta_i)}^{(i)} = \sum_{i=1}^{k} (\alpha_i + \beta_i) * c_{\sigma(\alpha_i + \beta_i)}^{(i)}. \tag{24}$$

Multiplication by scalars is given by

$$\gamma * a = \sum_{i=1}^{k} |\gamma|\alpha_i * c_{\sigma(\alpha_i)}^{(i)} = \sum_{i=1}^{k} |\gamma|\alpha_i * c_{\sigma(|\gamma|\alpha_i)}^{(i)}. \tag{25}$$

To every $a = \sum_{i=1}^{k} \alpha_i * c_{\sigma(\alpha_i)}^{(i)} \in \mathcal{S}$ we associate an the vector $(\alpha_1, \alpha_2, ..., \alpha_k)$. Then, minding formulae (24), (25), we define addition and multiplication by scalars by means of (7), (8), arriving thus to the canonic symmetric quasivector space $\mathbb{S}^k = (\mathbb{R}^k, +, \mathbb{R}, *)$ considered in Example 2.

As we know, negation in $\mathcal{S}$ is the same as identity. Conjugation in $\mathcal{S}$ coincides with opposite: $a_- = \mathrm{opp}(a) = \sum_{i=1}^{k} \alpha_i * c_{-\sigma(\alpha_i)}^{(i)} = \sum_{i=1}^{k} (-\alpha_i) * c_{\sigma(-\alpha_i)}^{(i)}$. This implies in terms of $\mathbb{S}^k = (\mathbb{R}^k, +, \mathbb{R}, *)$, cf. (9):

$$(\alpha_1, \alpha_2, ..., \alpha_k)_- = \mathrm{opp}(\alpha_1, \alpha_2, ..., \alpha_k) = (-\alpha_1, -\alpha_2, ..., -\alpha_k).$$

Theorem 4. *Any symmetric quasivector space over the l. o. field of reals $\mathbb{R}$, with a basis of k elements, is isomorphic to $\mathbb{S}^k = (\mathbb{R}^k, +, \mathbb{R}, *)$.*

Proof. Let $\mathcal{S}$ be a symmetric quasivector space spanned over a finite basis $s^{(1)}, s^{(2)}, ..., s^{(k)}$. The linear mapping $\varphi : \mathbb{S}^k \longrightarrow \mathcal{S}^k$, $\mathbb{S}^k = (\mathbb{R}^k, +, \mathbb{R}, *)$, defined by

$$\varphi(\alpha_1, \alpha_2, ..., \alpha_k) = \alpha_1 * s_{\sigma(\alpha_1)}^{(1)} + \alpha_2 * s_{\sigma(\alpha_2)}^{(2)} + ... + \alpha_k * s_{\sigma(\alpha_k)}^{(k)},$$

is a bijection. Hence φ is an isomorphism. $\square$

Let $\mathcal{S}$ be a symmetric quasivector space spanned over a finite basis $s^{(1)}, s^{(2)}, ..., s^{(k)}$. As in the linear case, the number k of terms in the expression for the span does not change with the particular basis, hence will be called *dimension* of $\mathcal{S}$.

As every quasivector space $\mathcal{Q}$ is a direct sum $\mathcal{Q} = \mathbb{V} \oplus \mathcal{S}$ of a vector space $\mathbb{V}$ and a symmetric quasivector space $\mathcal{S}$, we can speak of basis and dimension of $\mathcal{Q}$, whenever $\mathbb{V}$ and $\mathcal{S}$ have finite bases. Namely, let $\mathbb{V} = \mathbb{V}^l$ be a l-dimensional vector space with a basis $(v^{(1)}, ..., v^{(l)})$ and let $\mathcal{S} = \mathbb{S}^k$ be a k-dimensional symmetric quasivector space having a basis $(s^{(1)}, ..., s^{(l)})$. Then we say that $(v^{(1)}, ..., v^{(l)}; s^{(1)}, ..., s^{(l)})$ is a basis of the (l,k)-*dimensional* quasivector space $\mathcal{Q} = \mathbb{V}^l \oplus \mathbb{S}^k$.

Assume that we have a problem formulated in a quasilinear space of monoid structure $(\mathcal{M}, +, \mathbb{R}, *)$ — one may think of $\mathcal{M} = \mathcal{K}$ or $\mathcal{M} = I(\mathbb{R})$ — the set of intervals on $\mathbb{R}$. To reformulate the problem in the induced quasivector space $(\mathcal{D}(\mathcal{M}), +, \mathbb{R}, *)$ we first represent all elements $A \in \mathcal{M}$ involved as proper elements of $\mathcal{D}(\mathcal{M})$ of the form $a = (A, 0)$. Then, using Theorem 3 we decompose the problem into one linear problem and one symmetric quasivector problem. To this end we represent the element $a = (A, 0)$ in the form $a = a' + a'' = (a'; a'')$, wherein $a' = (1/2) * (a + a_-) = (1/2) * (A, \neg A)$; $a'' = (1/2) * (a \neg a) = (1/2) * (A \neg A, 0)$. We thus arrive to one linear problem and one symmetric quasivector problem, which are to be solved by means of usual techniques. What remains is the interpretation of the results in the original space of monoid structure $\mathcal{M}$ which depends on the particular problem.

Concluding Remarks. We have shown that any quasivector space is a direct sum of a linear subspace and a symmetric quasivector subspace. In the case of a finite basis the latter is isomorphic to the canonic symmetric quasivector space $(\mathbb{S}^k, +, \mathbb{R}, *)$. These results allow us to decompose any algebraic problem in a quasivector space into two problems: a linear problem and a problem in $(\mathbb{S}^k, +, \mathbb{R}, *)$. However, this simple results hold only for quasivector spaces and not for quasilinear spaces with monoid structure. It will probably cost a lot of intellectual effort before we learn how to formulate and interpret results in quasivector spaces having in mind that the natural problem formulation is in a quasilinear space with monoid structure.

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