MARTINGALE DECOMPOSITIONS AND INCOMPLETE MARKETS

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Abstract

We are interested in the problem of finding an optimal strategy for a non-attainable contingent claim in an incomplete market consisting of two traded assets. We discuss and compare the Kunita-Watanabe and Föllmer-Schweizer martingale decomposition and use them for finding the risk-minimizing trading strategy. The theory of stable spaces and minimal martingale measure is used and the differences in the models are discussed.

Key words: Incomplete markets, Martingale decompositions, Kunita-Watanabe and Föllmer-Schweizer decomposition

I. Introduction

In an incomplete market non-attainable contingent claims exist, they carry an intrinsic risk and their price is not unique. The choice of the price depends on the approach to the hedging problem and the investor's preference towards the risk. The different equivalent martingale measures provide a range of possible prices of the contingent claim. Compared to the complete market model, there are different questions to be answered - how to define a measure for the risk, how to construct a trading strategy that minimizes the risk and which equivalent martingale measure to choose. Under special assumptions of the model, answers to these questions are given by the Kunita-Watanabe and Föllmer-Schweizer decompositions.

Here we will compare the proposed solutions, including the different assumptions for the models. First we introduce the basics of the models and then discuss the two decompositions. Kunita-Watanabe martingale decomposition can be used for finding the optimal, risk minimizing strategy, when the discounted price process is a square-integrable martingale under the basic measure $\mathbb{P}$. When the price process is only a semimartingale, under additional assumptions the Föllmer-Schweizer decomposition can be used. The discussion on the minimal martingale measure connects the two decompositions.

II. Basics of the model

We observe a market consisting of 2 traded assets. One is riskless asset (bond) whose price process B_t is assumed to be strictly positive. The other is risky asset (stock) and its change of price is modeled by a stochastic process X_t with continuous paths. The processes are constructed on a probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ with a filtration satisfing the usual conditions. Here $\mathcal{F}_T = \mathcal{F}$, T being the time of maturity, and the σ-algebra $\mathcal{F}_t$ represents the information observable at time t. Of our main interest is the discounted price process $\frac{X_t}{B_t}$, so we will assume that $B \equiv 1$ and have in mind that X_t is notation for the discounted price process.

For a trading strategy $\phi = (\alpha, \beta)$, $(\alpha_t)_{t\in[0,T]}$ and $(\beta_t)_{t\in[0,T]}$ [1] will be the amounts invested at time t in the stock and the bond respectively. By definition, α is a predictable process with $\alpha \in L^2(\mathbb{P}_X)$ and β is in general an addapted process.[2] The *value process* of this portfolio is given by

$$V_t = \alpha_t X_t + \beta_t$$

and it is right-continuous with $V_t \in L^2(\mathbb{P})$. The *accumulated cost* process C_t is defined with

$$C_t = V_t - \int_0^t \alpha_s dX_s$$

where the second term in the above equation represents the accumulated gain by time t.

For a contingent claim, modeled here as an $\mathcal{F}_T$-measurable random variable *H*, $H \in L^2(\Omega, \mathcal{F}_T, \mathbb{P})$, we are looking for self-financing strategies[3] ϕ that replicate *H*, i.e. $V_T = H$ $\mathbb{P}$ a.s. Such a strategy is called *H-admissible* and if it exists *H* is an *attainable* contingent claim.

The question of replicating a contingent claim is answered in a complete market. In that model any contingent claim is attainable and this is equivalent to the martingale property of the market and the exsistance of a unique martingale measure ([5]). A replicating strategy can be found with the help of this measure. We are interested in redefining this question when the contingent claim is non-attainable. We need a new definition of an optimal strategy, since now any strategy carries a risk. We follow the discussions from [3], [6], [7] and [8].

III. Kunita-Watanabe decomposition

We work in an incomplete market model and we assume that the price process *X* is a square-integrable martingale under the basic probability measure $\mathbb{P}$. Let $H \in L^2(\mathbb{P})$ be a non-attainable contingent claim. We are looking for *H*-admisible strategies. Self-financing replicating strategies for *H* do not exist (as in the complete market case), so we have to define a way of choosing the optimal *H*-admisible strategy. The following risk-minimizing approach was proposed by Föllmer and Sondermann ([9]).

For a strategy $\phi = (\alpha, \beta)$ with a square-integrable cost process $C_t = V_t + \int_0^t \alpha_s dX_s$, we can define the *conditional*

[1] In the rest of the paper, for all the processes $t \in [0, T]$, unless otherwise specified.

[2] For more explanation of the meanings of the model assumptions, we refer to [1], [2], [3] and [4].

[3] For these strategies the cost process is constant $\mathbb{P}$ a.s. and the total wealth depends only on the initial investment and the stock price changes.

mean-square error process

$$R_t^\phi = \mathbb{E}((C_T - C_t)^2|\mathcal{F}_t), \qquad (1)$$

This process measures only the remaining cost till the time of maturity.

Every *H*-admisible risk minimizing strategy is mean self-financing[4] and has a value process which is a square-integrable martingale. Therefore the same holds for the integral $\int_0^t \alpha_s dX_s$ and the previous definition makes sence. This is why we will look for an optimal strategy only between the mean self-fininicing strategies.

A strategy $\psi = (\xi, \eta)$ is called *an admissible continuation of* ϕ at time t_0, if ϕ coincides with ψ at any time before t_0 and also $V_T^\phi = V_T^\psi$ at the maturity T. It will be *risk-minimizing* if at any time $t \in [0, T)$, $R_t^\phi \le R_t^\psi$, for any admissible continuation ψ of ϕ at time t.

It can be shown that a unique risk-minimizing strategy does exist and it can be constructed with the help of Kunita-Watanabe decomposition. We introduce some notations and part of the theory on stable spaces which we need for the main result. Details can be found in [6], [10] and [11].

Let $\mathcal{M}^2$ be the Hilbert space of all L^2-martingales on $(\Omega, (\mathcal{F}_t)_{t\ge0}, \mathcal{F}, \mathbb{P})$. For $M, N \in \mathcal{M}^2$ we say that are *weakly orthogonal* if $\mathbb{E}(M_\infty N_\infty) = 0$, and *strongly orthogonal* (notation $M \underline{\perp} N$) if their product MN is a (uniformly integrable) martingale. The last is equivalent to the quadratic variation process $\langle M, N\rangle$ being a (uniformly integrable) martingale.

For $F \subset \mathcal{M}^2$ we use the following notations

$$F^\perp = \{M : M \in \mathcal{M}^2, M \perp F\}$$

and

$$F^{\underline{\perp}} = \{M : M \in \mathcal{M}^2, M \underline{\perp} F\}$$

For any $F \subset \mathcal{M}^2$, $F^{\underline{\perp}}$ turns out to be a *stable space* i.e. closed and invariant under stopping. Even more, if *F* is a stable subspace of $\mathcal{M}^2$, then $F^\perp = F^{\underline{\perp}}$. Therefore, for a closed subspace *F* of the inner product space $\mathcal{M}^2$, we have the decomposition $\mathcal{M}^2 = F \oplus F^{\underline{\perp}}$.

The following theorem gives the Kunita-Watanabe decomposition of a square-integrable martingale $M \in \mathcal{M}^2$. With

$$\mathcal{S}(F) = \cap\{G : G \subseteq \mathcal{M}^2, \text{ G is stable and } F \subseteq G\}$$

we will denote the stable subspace generated by *F*.

Theorem 1. *Let* $F = \{X^1, ..., X^n\} \subseteq \mathcal{M}^2$ *where* $X^i \underline{\perp} X^j$, *for* $i \ne j$. *For any martingale* $X \in \mathcal{M}^2$ *we define*
$\mathcal{A}_X = \{\alpha : \alpha \text{ is predictable and } \mathbb{E}(\alpha^2 \cdot \langle X\rangle) < \infty\}$
Then
a) $\mathcal{S}(F) = \{\sum_{i=1}^n (\alpha^i \cdot X^i) : \alpha^i \in \mathcal{A}_{X^i}\}$,[5]
b) Any $M \in \mathcal{M}^2$ *can be decomposed as*

$$M_t = M_0 + \underbrace{\sum_{i=1}^n \int_0^t \alpha_s^i dX_s^i}_{\in \mathcal{S}(F)} + \underbrace{L_t}_{\in \mathcal{S}(F)^{\underline{\perp}}} \qquad (2)$$

[4] The cost process is a $\mathbb{P}$-martingale.
[5] *The notation* $\alpha \cdot X$ *represents the stochastic integral* $\int_0^\cdot \alpha_s dX_s$

The theorem gives a characterisation of the stable space $\mathcal{S}(F)$ - each element of $\mathcal{S}(F)$ can be represented as a sum of stochastic integrals with respect to the elements of *F*. Since $\mathcal{M}^2 = \mathcal{S}(F) \oplus \mathcal{S}(F)^{\underline{\perp}}$, any martingale has the *Kunita-Watanabe decomposition* (2).

The component L_t here is orthogonal to all the integrals in the decomposition.

We go now back to the market model, where the non-attainable contingent claim *H* defines the square-integrable martingale $H_t = \mathbb{E}(H|\mathcal{F}_t)$. Since we have assumed that the discounted price process *X* is also a square-integrable martingale, we can apply the previous theorem and decompose H_t as

$$H_t = \mathbb{E}(H) + \int_0^t \alpha_s^H dX_s + L_t^H \qquad (3)$$

Let us now define the strategy $\phi^H = (\alpha^H, V^H - \alpha^H X)$. Here $V_t^H = H_t$, $t \in [0, T]$ is the value process of the strategy. Obviously this is an *H*-admissible strategy. Since the cost process $C_t^H = \mathbb{E}(H) + L_t^H$ is a martingale, this is a mean self-fininicing strategy. Let $R_t^\phi = R_t^H = \mathbb{E}((L_T^H - L_t^H)^2|\mathcal{F}_t)$ denote the intrinsic risc of *H* and also, let $\psi = (\alpha, \beta)$ be some admissible continuation of ϕ^H for some $t < T$. It can be shown that

$$R_t^\psi = \mathbb{E}(\int_t^T (\alpha_s^H - \alpha_s)^2 d\langle X\rangle_s) + (V_t^H - V_t)^2 + R_t^H$$

R_t^ψ is minimized when $\alpha_s = \alpha_s^H$ on [t,T] and $V_t = V_t^H$. But since t is arbitraty on [0,T], $\alpha = \alpha^H$ and $V = V^H$ on [0,T]. Therefore $R_t^\phi \le R_t^\psi$ and ϕ^H is the unique admissible risk-minimizing strategy.

IV. Föllmer-Schweizer decomposition

In the previous discussion we have assumed that the processes *X* and *H* are square-integrable martingales under the basic probability measure $\mathbb{P}$. In this part we assume that *X* is just a semi-martingale under $\mathbb{P}$, with a semi-martingale decomposition *X=M+A*. Additionaly we assume that *M* is a square-integrable martingale under $\mathbb{P}$. For a contingent claim *H* we are searching for strategies $\phi = (\alpha, \beta)$ with a value process satisfying $V_T = H$. The process *V* is not a martingale, so we can not apply the results from the previous discussion.

The cost process of the strategy is given by $C_t = V_t - \int_0^t \alpha_s dM_s - \int_0^t \alpha_s dA_s$. Now the last term also influences the risk-process R^ϕ.

We say that an admissible strategy $\phi = (\alpha, \beta)$ is *locally risk-minimizing*[6] or *optimal* if its cost process C_t is a square-integrable martingale, orthogonal to the process *M* under $\mathbb{P}$. For such a strategy, we have

$$H = V_T = H_0 + \int_0^T \alpha_s dX_s + L_T \qquad (4)$$

with $H_0 = C_0$, $L_t = C_t - C_0 \in \mathcal{M}^2$, $L \underline{\perp} M$ and $\alpha \in \mathcal{A}_X$. This decomposition is called *Föllmer-Schweizer decomposition*. We see that it is similar to (2). Still, here *X* is no longer a square-integrable martingale and *L* is orthogonal only to the

[6] See [7] on details.

martingale component of X , insteed of X itself. We can try to work under an equivalent martingale measure but eventhough X will become a square-integrable martingale, the process L in (4) will change and it may no longer be a martingale.

For a contingent clain H, which has the decomposition (4), we can define $V_t = H_0 + \int_0^t \alpha_s dX_s + L_t$ and a strategy $\phi = (\alpha, V - \alpha X)$. The cost process of this strategy satisfies $C_t = H_0 + L_t$, so $C \in \mathcal{M}^2$ and $C \underline{\underline{\perp}} M$. Therefore, the decomposition of H (4) is equivalent to the existance of a locally risk-minimizing strategy. In [12] the last decomposition and the optimal strategy were connected to the concept of the minimal martingale measure. An equivalent martingale measure $\mathbb{Q}^*$ is called *minimal martingale measure*, if any $L \in \mathcal{M}^2(\mathbb{P})$ such that $L \underline{\perp} M$ under $\mathbb{P}$, is also $\mathbb{Q}^*$-martingale. When this measure exists the two decompositions coincide and the optimal strategy can be found as before. We present here the main result from [12] and we will then be able to make the connection between the two decompositions. More details can be also found in [8].

For the semimartingale X, we consider the equivalent martingale measures $\mathbb{Q}$ with square-integrable densities $Z_t = \mathbb{E}_{\mathbb{P}}(Z_T|\mathcal{F}_t)$, where $Z_T = \frac{d\mathbb{Q}}{d\mathbb{P}}$ is the Radon-Nikodym derivative. Girsanov theorem gives us the semimartingale representation of the $\mathbb{Q}$-martingale $X = M^{\mathbb{Q}}$ under $\mathbb{P}$

$$X_t = M_t^{\mathbb{Q}} = M_t + A_t$$

We will assume that $M \in \mathcal{M}^2(\mathbb{P})$. The processes X and Z can be decomposed as

$$X_t = M_t + \int_0^t \eta_s d\langle M \rangle_s \qquad (5)$$

and

$$Z_t = 1 - \int_0^t \eta_s Z_s dM_s + N_t \qquad (6)$$

with $\eta \in \mathcal{A}_M$, $N \in \mathcal{M}^2(\mathbb{P})$ and $\langle N, M \rangle_t = 0$ (N $\underline{\perp}$M).

(6) is a characteristical decomposition of all the square-integrable martingale densities. We are interested when (6) with $N_t = 0$ a.s. is still a density of an equivalent martingale measure. We use the notation of Doléans-Dade exponential $\mathcal{E}(X)$ for the solution of the SDE $Z_t = 1 + \int_0^t Z_{s-} dX_s$. Combining Girsanov's theorem and Novikov's condition, we get the answer with the following theorem.

Theorem 2. *Let X_t be the continuous semimartingale given with* (5) *and let $K_t = -\int_0^t \eta_s dM_s$ for $t \in [0,T]$. If $Z_t^* = \mathcal{E}(K)_t = exp(K_t - \frac{1}{2}\langle K \rangle_t)$ is uniformly integrable, then it is a density process for an equivalent martingale measure $\mathbb{Q}^*$.*

This process is directly connected to the minimal martingale measure.

Theorem 3. *a) The existance of the minimal martingale measure $\mathbb{Q}^*$ is equivalent to the square-integrability of the process $Z^* = \mathcal{E}(-\int_0^\cdot \eta_s dM_s)$. If it exists it is unique and is determined by $Z_T^* = \frac{d\mathbb{Q}^*}{d\mathbb{P}}$;*

b) $\mathbb{Q}^*$ preserves orthogonality, i.e. if $L \in \mathcal{M}^2(\mathbb{P})$ and $\langle L, M \rangle = 0$, then $\langle L, X \rangle = 0$ under $\mathbb{Q}^*$.

We see that the only information that we need to construct the minimal martingale measure is the information about the process X. From its semimartingale decomposition we determine the process M and we can check whether the process $\mathbb{Z}^*$ is square-integrable. If it is, $\mathbb{Z}_T^*$ is the Radon-Nikodym derivative of the measure $\mathbb{Q}^*$. Let now H be a contingent claim $H \in L^2(\mathbb{P})$. Then, $Z^* \in L^2(\mathbb{P})$ implies that $H \in L^1(\mathbb{Q}^*)$. We assume that the process $H = \mathbb{E}_{\mathbb{P}}(H|\mathcal{F}_t)$ has the Föllmer-Schweizer decomposition (4) $\quad H_t = H_0 + \int_0^t \alpha_s dX_s + L_s$. Here L is orthogonal only to the martingale component M, of the semimartingale decomposition of the process X=M+A. Also, $L \in \mathcal{M}^2(\mathbb{P})$ and since $\mathbb{Q}^*$ preserves the orthogonality, we have $\langle L, X \rangle = 0$, i.e. L is orthogonal also to X. Therefore, this is the Kunita-Watanabe decomposition of H under $\mathbb{Q}^*$. Of course, if H is square-integrable under $\mathbb{Q}^*$ then the composition will anyway exist.

With this we have seen that, if the contingent claim can be decomposed as in (4) and if the measure $\mathbb{Q}^*$ does exist, (4) actually will represent the Kunita-Watanabe decomposition under $\mathbb{Q}^*$. Once we conclude this, we can proceed as in the first case, and find the optimal strategy. This gives the connection between the two decompositions and aslo solves the problem of finding the optimal strategy.

REFERENCES

[1] H.FÖLLMER, *Probabilistic Aspects of Options*, Series:Sonderforshungsbereich 303, Creation-Date: 1991-11

[2] R.KORN, E.KORN, *Option Pricing and Portfolio Optimization*, Vol. 31 American Mathematical Society, 2000

[3] M.SCHWEIZER, *Option Hedging for Semimartingales*, Stochastic Processes and their Applications 37 (1991), 339-363

[4] W.SCHACHERMAYER, *Introduction to the Mathematics of Financial Markets*, In: S. Albeverio, W. Schachermayer, M. Talagrand: Lecture Notes in Mathematics 1816 - Lectures on Probability Theory and Statistics, Saint-Flour summer school 2000 (Pierre Bernard, editor), Springer Verlag, Heidelberg, (2003), pp. 111-177

[5] J.M. HARRISON, S.R. PLISKA, *A Stochastic Calculus Model of Continuous Trading: Complete Markets*, Stochastic Process. Appl. 15, 313-316 (1983) "

[6] H.KUNITA, S.WATANABE, *On Square-Integrable Martingales*, Nagoya Math. J. Volume 30, (1967), 209-245

[7] M.SCHWEIZER, *A Guided Tour through Quadratic Hedging Approaches*, E. Jouini, J. Cvitanic, M. Musiela (eds.), "Option Pricing, Interest Rates and Risk Management", Cambridge University Press,(1999) 538-574

[8] B.TOJTOVSKA, *Martingale Decompositions with Application in Finance*, Master Thesis, TU Kaiserslautern 2007

[9] H.FÖLLMER, D.SONDERMAN, *Hedging of Non-redundant Contingent Claims*, W. Hildebrand and A. Mas-Collel, (Eds.), Contributions to Mathematical Economics in Honor of Gerard Debreu, Elsevier NorthHolland, Amsterdam,(1986), pp. 205-223.

[10] P.PROTTER, *Stochastic Integration and Differential Equations*, 2nd Ed. Springer, 2004

[11] I.KARATZAS, S.SHREVE, *Brownian Motion and Stochastic Calculus*, Edition Springer-Verlag, 1988

[12] M.SCHWEIZER, *On the Minimal Martingale Measure and the Föllmer-Schweizer Decomposition*, Stochastic Analysis and Applications 13, (1995), p. 573-599