An upper asymptotic bound of the mean square worst-case error of the integration in the weighted Sobolev space by using "tent transformation"

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ABSTRACT

In this paper we consider the mean square worst-case error of the integration in the weighted Sobolev space introduced by Hickernell.

We approximate the integrals of integrands from this space through quasi-Monte Carlo algorithm with equal quadrature weights. It is obtained an exact formula for the mean square worst-case error of the integration in the space as a Fourier-Walsh series.

It is introduced a new transformation of the points of nets in $[0,1]^s$ the so-called a tent transformation in base b, where b is an arbitrary even number.

We obtain an upper bound of the mean square-worst case error of the integration in this space, by using first points of (s,b)-digitally shifted and second folded by the tent transformation in base b, an arbitrary (t,m,s)-net. This permits us to give the asymptotic behaviour of this error.

I. PRELIMINARY NOTATIONS AND STATEMENTS

Let $s\geq 1$ be a fixed integer and $\xi=(\boldsymbol{x}_i)_{i\geq 0}$ be an arbitrary sequence of points in $[0,1)^s$. For each integer $N\geq 1$ and an arbitrary subinterval J of $[0,1)^s$ with a volume $\mu(J)$, we denote by $A(\xi,J,N)$ the number of the points $\boldsymbol{x}_n$ of the sequence ξ whose indices n satisfy the inequalities $0\leq n\leq N-1$ and belong to the interval J. The sequence ξ is called **uniformly distributed** in $[0,1)^s$ if the equality

$$\lim_{N\to\infty}\frac{A(\xi,J,N)}{N}=\mu(J)$$

holds for every subinterval J of $[0,1)^s$.

The **diaphony** is a quantitative measure for uniform distribution of sequences in $[0,1)^s$. Zinterhof [15] uses the trigonometric functional system $\boldsymbol{\tau}$ to introduce the "classical" diaphony

$$\boldsymbol{\tau}=\{e_{\mathbf{m}}(\mathbf{x})=\exp(2\pi I(\mathbf{m},\mathbf{x})):\mathbf{m}=(m_1,m_2,...,m_s)\in\mathbf{Z}^s,$$

$\mathbf{x}=(x_1,x_2,...,x_s)\in[0,1)^s\}$, where $(\mathbf{m},\mathbf{x})$ is the inner product between the vectors $\mathbf{m}$ and $\mathbf{x}$.

Weyl criterion:

The sequence $\xi=(\boldsymbol{x}_i)_{i\geq 0}$ of points in $[0,1)^s$ is uniformly distributed if and only if the limit equality

$$\frac{1}{N}\lim_{N\to\infty}\sum_{n=0}^{N-1}e_m(x_n)=0$$

holds for each integer vector $\mathbf{m}\neq\mathbf{0}$.

For a non-negative integer $k\geq 0$ and real $x\in[0,1)$ with the b-adic representations

$$k=\sum_{i=0}^{\upsilon}k_i b^i \text{ and } x=\sum_{i=0}^{\infty}x_i b^{-i-1},$$

where for $0\leq i\leq\nu$, $k_i\in\{0,1,\dots,b-1\}$, $k_\nu\neq 0$, the k-th **function of Walsh in base b** ${}_b wal_k(x):[0,1)\to\mathbf{C}$ is defined in the following way

$${}_b wal_k(x)=\prod_{i=0}^{\upsilon}\exp\left(2\pi I\frac{k_i x_i}{b}\right).$$

Let N_0 be the set of non-negative integers, for an arbitrary vector $\boldsymbol{k}=(k_1,...,k_s)\in N_0^s$ and $\boldsymbol{x}=(x_1,...,x_s)\in[0,1)^s$ we define the multivariate Walsh function by multiplication of the corresponding univariate functions, i.e.

$${}_b wal_{\boldsymbol{k}}(\boldsymbol{x})=\prod_{j=1}^{s}{}_b wal_{k_j}(x_j).$$

The set $W(b)=\{{}_b wal_k(\mathrm{x}):k=0,1,\dots;x\in[0,1)\}$ is called the Walsh functional system in base b.

Weyl criterion:

The sequence $\xi=(\boldsymbol{x}_i)_{i\geq 0}$ of points in $[0,1)^s$ is uniformly distributed if and only if the limit equality

$$\frac{1}{N}\lim_{N\to\infty}\sum_{n=0}^{N-1}{}_b wal_k(x_n)=0$$

holds for each integer vector $\boldsymbol{k}\in N_0^s$ and $\mathbf{k}\neq\mathbf{0}$.

Grozdanov [12] in 2007 introduced the so-called **weighted b-adic diaphony**. So, let $\alpha>1$ be an arbitrary real and γ be an arbitrary vector of positive weights $\gamma=(\gamma_1,\dots,\gamma_s)$ where $\gamma_1\geq\gamma_2\geq\dots\geq\gamma_s>0$. For each integer $N\geq 1$ the **weighted b-adic diaphony** $F_N(\boldsymbol{W(b)},\alpha,\gamma,\boldsymbol{\xi})$ of the first N elements of the sequence $\xi=(\boldsymbol{x}_i)_{i\geq 0}$ of points in $[0,1)^s$ is defined as

$$F_N(W(b),\alpha,\gamma,\xi)=$$
$$=\sqrt{\frac{1}{C(b,\alpha,\gamma)-1}\sum_{\boldsymbol{k}\in N_0^s,\boldsymbol{k}\neq\boldsymbol{0}} r_b(\alpha,\gamma,\boldsymbol{k})\left|\frac{1}{N}\sum_{n=o}^{N-1} {}_b wal_k(\boldsymbol{x}_n)\right|^2},$$

where for a vector $\boldsymbol{k}=(k_1,...,k_s)\in \boldsymbol{N}_0^s$, $r_b(\alpha,\gamma,\boldsymbol{k})=\Pi_{j=1}^s r_b(\alpha,\gamma_j,k_j)$, and for each integer $k\geq 0$ the coefficient $r_b(\alpha,\gamma_j,k_j)$ is defined by:

$$r_b(\alpha,\gamma;k)=\begin{cases}1; if\ k=0\\ \gamma b^{-\alpha g}; for\ \forall k, b^g\leq k<b^{g+1}, g\in \boldsymbol{N}_0\end{cases}$$

where $C(b,\alpha,\gamma)=\Pi_{j=1}^s[1+\gamma_j\mu_b(\alpha)]$ and

$$\mu_b(\alpha)=\frac{(b-1)b^\alpha}{b^\alpha-b}.$$

Theorem 1:
The sequence $\xi=(\boldsymbol{x}_i)_{i\geq 0}$ of points in $[0,1)^s$ is uniformly distributed if and only if the equality

$$\lim_{N\to\infty} F_N(W(b),\alpha,\gamma,\xi)=0,$$

for each real $\alpha>1$ and for each vector of positive weights $\boldsymbol{\gamma}$ holds.

II. Multivariate integration in weighted Hilbert spaces

Following Aronszajn [8], we will recall the concept of reproducing kernels for Hilbert spaces.
So, let $H_s(K)$ be a Hilbert space with a reproducing kernel K $:[0,1]^{2s}\to\mathbf{R}$ and a norm $\|\cdot\|_{H_s(K)}$.
The multivariate integral

$$I_s(f)=\int_{[0,1]^s} f(x)dx,\ f\in H_s(K)$$

is approximated by quasi-Monte Carlo algorithm with equal quadrature weights

$$Q_s(f;P_N)=\frac{1}{N}\sum_{n=0}^{N-1} f(x_n),$$

where $P_N=\{x_0, x_1...,x_{N-1}\}$ is a deterministic sample point net composed of $N\geq 1$ points in $[0,1)^s$.
The worst-case error of the integration in the space $H_s(K)$ is defined as

$$e(H_s(K);P_N)=\sup_{f\in H_s(K),\|f\|_{H_s(K)}\leq 1} |I_s(f)-Q_s(f;P_N)|.$$

For arbitrary reals $x,y\in[0,1)^s$ with representations

$$x=\sum_{i=0}^\infty x_i b^{-i-1}$$

and

$$y=\sum_{i=0}^\infty y_i b^{-i-1}$$

where for infinitely many indices i, $x_i, y_i\neq b-1$, let us set

$$x\oplus_1^b y=\sum_{i=0}^\infty [x_i+y_i(\bmod b)]b^{-i-1}.$$

For arbitrary vectors $\mathbf{x}=(x_1,x_2,...,x_s)$ and $\mathbf{y}=(y_1,y_2,...,y_s)\in[0,1)^s$ let us define $\boldsymbol{x}\oplus_s^b\boldsymbol{y}=(x_1\oplus_1^b y_1,...,g_s\oplus_1^b y_s)$.
Let

$$P_N=\{x_0, x_1...,x_{N-1}\}$$

be an arbitrary net in $[0,1)^s$. For an arbitrary vector $\sigma\in[0,1)^s$ we define the so-called **(s,b)-digitally shifted net** $P_N((s,b);\sigma)$ by

$$P_N((s,b);\sigma)=\{x_0\oplus_s^b\sigma, x_1\oplus_s^b\sigma,..., x_{N-1}\oplus_s^b\sigma\}.$$

Following Dick and Pillichshammer [5] we recall the definition for mean square worst-case error of the integration in Hilbert spaces. Let $H_s(K)$ be an arbitrary Hilbert space generated by the kernel K. Let P_N be an arbitrary net composed of N points in $[0,1)^s$. We define the notion of a

mean square worst-case error $\hat{e}_{(s,b)-ds}(H_s(K);P_N)$

of the integration in the space $H_s(K)$ by using a random *(s,b)*-digitally shifted net P_N by the equality

$$\hat{e}_{(s,b)-ds}(H_s(K);P_N)=$$
$$\left(\int_{[0,1]^s} e^2(H_s(K);P_N((s,b);\sigma))d\sigma\right)^{\frac{1}{2}}.$$

III. Multivariate integration in the weighted Sobolev space

In our work we will realize an investigation of the mean square worst-case error of the integration in the space H_{Sob,s,γ,β_4} of functions which partial derivatives up to order two have to be square integrable. This space have been introduced by Hickernell [1].
Let B_k denotes the k-th Bernoulli polynomial, i. e. $B_0(x)=1$, $B_1(x)=x-1/2$, $B_2(x)=x^2-x+1/6$ and $B_4(x)=x^4-2x^3+x^2-1/30$.
Let us denote $\beta_4=\{B_0,B_1,B_2,B_4\}$.
Hickernell [1] has defined the Sobolev space

$$H_{Sob,s,\gamma,\beta_4}=\{h: \|h\|<\infty\}.$$

The reproducing kernel of the space H_{Sob,s,γ,β_4} is given by

$$K_{s,\gamma}(\mathbf{x},\mathbf{y})=\Pi_{j=1}^s K_{\gamma_j}(x_j,y_j),$$

$\mathbf{x}=(x_1,x_2,...,x_s)$ and $\mathbf{y}=(y_1,y_2,...,y_s)\in[0,1)^s$, where for reals x, y in [0,1) and a weight $\gamma>0$ the one-dimensional reproducing kernel $K_\gamma(x, y)$ is defined as

$$K_\gamma(x,y)=$$
$$B_0(x)B_0(y)+\gamma B_1(x)B_1(y)+\gamma^2/4B_2(x)B_2(y)-\gamma^2/24B_4(|x-y|)$$

Theorem 2:
The mean square worst-case error of the integration in the weighted Sobolev space by H_{Sob,s,γ,β_4} using a random *(s,b)*-digitally shifted $P_N=\{x_0, x_1...,x_{N-1}\}$ is given by the equality

$$\hat{e}^2_{(s,b)-ds}(H_{Sob,s,\gamma,\beta_4};P_N)=$$
$$-1+\frac{1}{N^2}\sum_{n=0}^{N-1}\sum_{h=0}^{N-1}\sum_{k\in N_0^s}\hat{K}_{W(b),s,\gamma}(k,k)_b\, wal_k(x_n)\,\overline{{}_b wal_k(x_h)},$$

where for an arbitrary vector of weights $\gamma=(\gamma_1,...,\gamma_s)$ and a vector $\boldsymbol{k} \in \boldsymbol{N}_0^s$ the coefficients $\hat{K}_{W(b),s,\gamma}(k,k)$ are exactly obtained.

Let b≥2, s≥1 and 0≤t≤m be integers. A point set P consisting of b^m points in $[0,1)^s$ forms a **(t,m,s)-net in base *b*** if every subinterval J

$$J = \prod_{j=1}^{s}\left[\frac{a_j}{b^{d_j}}, \frac{a_j+1}{b^{d_j}}\right)$$

of $[0,1)^s$ with integers $d_j \geq 0$ and $0 \leq a_j \leq b^{d_j}$ for 1≤j≤s and of volume b^{t-m} contains exactly b^t points of the net P.

IV. Tent transformation

Let b≥2 be a given **even** integer. We will define the function:

$$\Phi(x) = \begin{cases} bx - 2\alpha, \; if \; x \in [\frac{2\alpha}{b}, \frac{2\alpha+1}{b}), \\ \alpha = 0,1,\ldots,\frac{b-2}{2} \\ 2 + 2\alpha - bx, \; if \; x \in [\frac{2\alpha+1}{b}, \frac{2\alpha+2}{b}), \\ \alpha = 0,1,\ldots,\frac{b-2}{2} \end{cases}$$

which we will call a tent transformation in base b.

For a vector $\mathbf{x}=(x_1,x_2,\ldots,x_s) \in [0,1)^s$ we set:

$$\Phi(\mathbf{x}) = (\Phi(x_1), \Phi(x_2), \ldots, \Phi(x_s)).$$

To the points of the net $P_N((s,b);\sigma)$ we apply the tent transformation and obtain the net

$$P_N((s,b);\Phi;\sigma)=\{\Phi(x_0) \oplus_s^b \sigma, \Phi(x_1) \oplus_s^b \sigma, \ldots, \Phi(x_{N-1}) \oplus_s^b \sigma\}.$$

Theorem 3:

Let $K \in L_2([0,1)^{2s})$ be an arbitrary reproducing kernel.

The mean square worst-case error $\hat{e}_{(s,b)-ds}(H_s(K);P_N)$ of the integration in the space $H_s(K)$ by using first (s,b)-digitally shifted and second folded by the tent transformation net P_N satisfies the equality

$$\hat{e}_{(s,b)-ds;\Phi}(H_s(K);P_N) = e(H_s(K_{(s,b)-ds;\Phi});P_N),$$

where

$$e(H_s(K_{(s,b)-ds;\Phi});P_N)$$

is the worst-case error of the integration in the Hilbert space, generated by the associated (s,b)-digitally shifted and folded kernel $K_{(s,b)-ds;\Phi}$.

Theorem 4 (an upper bound):

Let $1/4 < \lambda \leq 1$ be a given real number. There exists a digital (t,m,s)-net P_{b^m} over $\mathbf{Z}_b$, such that the mean square worst-case error of the integration in the space H_{Sob,s,γ,β_4} by using first (s,b)-digitally shifted and second folded by the tent transformation in base b net P_{b^m} satisfies the upper bound

$$\hat{e}^{2\lambda}_{(s,b)-ds,\Phi}(H_{Sob,s,\gamma,\beta_4};P_{b^m}) \leq$$
$$-1 + \prod_{j=1}^{s}[1 + c_1(b,\lambda,\gamma_j)\frac{1}{b^{4\lambda m}}] +$$
$$\frac{1}{b^m}\prod_{j=1}^{s}[1 + c_2(b,\lambda,\gamma_j) + c_3(b,\lambda,\gamma_j)\frac{b^m}{b^{4\lambda m}} +$$
$$+ c_4(b,\lambda,\gamma_j)\frac{1}{b^{4\lambda m}}],$$

where the $c_i(b,\lambda,\gamma_j)$, 1≤i≤4, 1≤j≤s are exactly defined constants.

Theorem 5 (an asymptotic behaviour):

The mean square worst-case error of the integration in the space H_{Sob,s,γ,β_4} by using first (s,b)-digitally shifted and second folded by the tent transformation in base b net P_{b^m} satisfies

$$\hat{e}_{(s,b)-ds,\Phi}(H_{Sob,s,\gamma,\beta_4};P_{b^m}) = O(b^{-2m})$$

V. References

[1] F. J. Hickernell "Obtaining O(N^(-2+ϵ)) Convergence for lattice Quadrature Rules", In: *Fang, K. T., Hickernell, F. J.,Niederreiter, H. (eds.) Monte Carlo* and *Quasi-Monte Carlo Methods 2000*, pp. 274-289, Springer, Berlin Heidelberg New York, 2002.

[2] G. Larcher, H. Niederreiter, W.Ch. Schmid "Digital nets and sequences *constructed* over finite rings and their applications to quasi-Monte Carlo integration", *Monatsh. Math.*, vol.121, no. 3, pp.231-253, 1996.

[3] G. Larcher and G. Pirsic, "Base change problems for generalized Walsh series and multivariate numerical integration", *Pacific J. Math,* vol.189, no.1, pp. 75-105, 1999.

[4] H. E. Chrestenson " A class of generalized Walsh functions", *Pacific J.* Math*,* vol.5, pp. 17-31, 1955.

[5] J. Dick, F. Pillichshammer, "Multivariate integration in weighted Hilbert spaces based on Walsh functions and weighted Sobolev spaces", *J. Complexity*, vol. 21, pp. 149-195, 2005.

[6] J. L. Walsh "A closed of normal orthogonal functions", *Amer J. Math,* vol.45, pp.5-24, 1923.

[7] L. Cristea, J. Dick, G. Leobacher, F. Pillichshammer "The tent transformation can improve the convergence rate of quasi-Monte Carlo algorithms using digital nets", *Numer. Math*. vol. 105 pp. 413-455, 2007

[8] N. Aronszajn "Theory of reproducing kernels", *Trans. Amer. Math. Soc.* Vol.68, pp.337-404, 1950.

[9] V. Grozdanov and S. Stoilova, "On the theory of the *b*-adic diaphony", *C. R. Acad. Bul. Sci.*, vol.54, no.3, pp. 31–34, Sophia, 2001.

[10] V. Ristovska, "Irregularity of the distribution of the generalized Van der Corput sequence*", C.R.Acad. Bul. Sci.*, vol.58, no.5, pp. 487–492, 2005.

[11] V.Ristovska, "Quantitative measures for uniform distribution of sequences*", PMF,* Skopje, 2007 (master thesis).

[12] V.S. Grozdanov, "The weighted b-adic diaphony", *Journal of Complexity*, vol. 22, 490-513, 2006.

[13] V. Grozdanov, E. Nikolova, S. Stoilova, "Generalized *b*-adic diaphony", *C. R. Acad. Bul. Sci.*, vol.56, no. 4, pp. 23–30, Sophia, 2003.

[14] P. Zinterhof "Uber einige Abschatzungen bei der Approximation von Funktionen mit Gleichverteilungsmethoden", *S. B. Akad. Wiss., math.-naturw. Klase, Abt.II,* vol.185, pp.121-132, 1976.