

THE STUDY OF ELLIPSES AND HYPERBOLAS AS SECOND ORDER CURVES USING GEOGEBRA 4

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ABSTRACT

In the third year of middle school (11th grade), pupils are introduced to circles, ellipses, hyperbolas and parabolas as second order curves or conic sections. These curves are not studied in depth. However, they have great significance in the arts and sciences and we can use this to increase interest.

In this paper, we demonstrate ways to give physical meaning to ellipses and hyperbolas. We then use the freeware GeoGebra 4.0 to virtualize and visualize this process. With this version there are greater possibilities for working with these curves. We can input all of their different definitions starting with the canonical (implicit) equations and then dynamically show how and why the definitions coincide. Finally, we demonstrate a project in which pupils/students use GeoGebra to build a simulator of a natural phenomenon that generates hyperbolas.

I. INTRODUCTION

Second order curves, namely parabolas, ellipses and hyperbolas occur in real life. We study parabolas as the graphs of quadratic functions and we spend a great deal of time on quadratic functions. However, we only briefly mention ellipses and hyperbolas and usually only theoretically. In this paper, we look at ways to connect ellipses and hyperbolas with real phenomena and to connect these with the various forms of the equations and by creating interactivities and simulators with GeoGebra 4.

II. ELLIPSES

A. Getting Started with Ellipses

There are many different ways of writing the equations and functions that yield an ellipse.

We start with the canonical form of the ellipse, namely:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \text{ where } a, b \in \mathbb{R}^+ \quad (1)$$

Exercise 1: GeoGebra and the Canonical Form of Ellipse

- Start GeoGebra and open Algebra View
- Create sliders for a and b as positive real numbers.
- In the Input field, input: $x^2/a^2 + y^2/b^2 = 1$. The ellipse will be drawn. In the Algebra view, by default, the equation is written in 2nd order curve form. Right-click on the equation of the ellipse and (a) notice that GeoGebra recognizes that this is an ellipse and (b) select the canonical form of the equation (Figure 1).
- Using the Intersect tool, create points for the intersections of the ellipse with the x -axis and for the intersections of the ellipse with the y -axis. Compare the x -value of the first point to a and the y -value of the second point to b (Figure 2).

- Change the values of a and b and check that this relationship remains true.

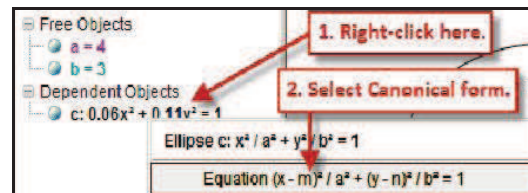


Figure 1: Switch between canonical and 2nd order form

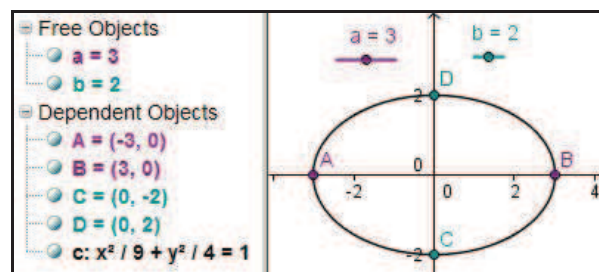


Figure 2: Canonical Form of Ellipse in GeoGebra

B. Explicit Form of an Ellipse

- We continue with our previous GeoGebra worksheet. Solve equation (1) for y . Input $f(x)$ as the positive root. $f(x) = b \cdot \sqrt{1 - x^2/a^2}$. Input $g(x) = -f(x)$. These graphs should coincide with the top and bottom of the graph of the ellipse in canonical form.
- Add colors and styles to the graphs and then add check boxes to turn on and off the Canonical and Explicit forms. Label the functions (see Figure 3).
- Use the Length[] and Integral[] commands with $f(x)$ to find the circumference and area of the ellipse. $c_e = 2 \cdot \text{Length}[f, -a, a]$ and $a_e = 2 \cdot \text{Integral}[f, -a, a]$. Find the formulas on the internet and compare results.

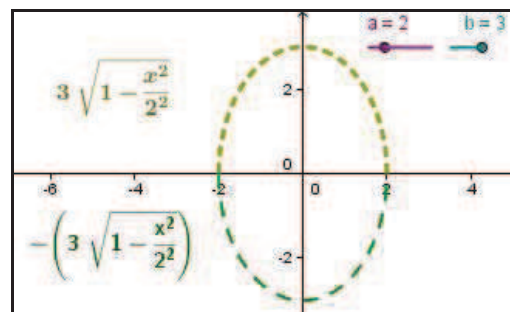


Figure 3: Explicit Equations for the Ellipse

C. Drawing an Ellipse – A Hands-on Experience

A simple, hands-on way of introducing ellipses is to have the pupil actually draw an ellipse by using 2 pushpins for the foci and a fixed length of string, paper, pencil and a board. This process is shown in Figure 4.

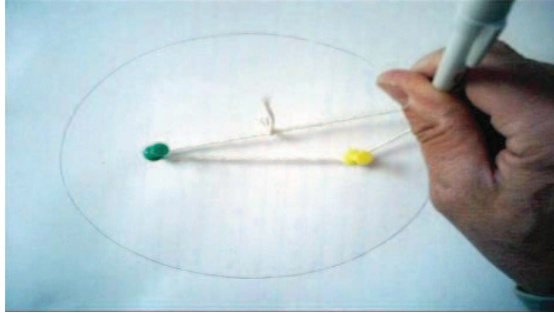


Figure 4: Drawing an Ellipse in Real Life [2]

Exercise 2: Draw an Ellipse and Relate it to Canonical

- Draw an ellipse with pushpins and string as above.
- Write F_1 on the right pushpin hole and F_2 on the left pushpin hole.
- Draw a line through the holes as the x -axis.
- Find the midpoint between the holes and draw a perpendicular through this point. It is the y -axis.
- Measure the distance between the pushpins: $d = \underline{\hspace{2cm}}$
- Measure the length of the string: $s = \underline{\hspace{2cm}}$
- Write an equation using s and d . For every point P on the ellipse, $|F_1P| + |F_2P| = \underline{\hspace{2cm}}$.
- Determine the coordinates of F_1 and F_2 .
- Find the values a and b from the intersection points of the ellipse and the x - and y - axes respectively.
- Determine the coordinates of these intersection points.
- Write an equation using a . For every point P on the ellipse, $|F_1P| + |F_2P| = \underline{\hspace{2cm}}$.
- Write an equation for a in terms of s and d and check it against your measured data. $a = \underline{\hspace{2cm}}$
- ** Challenge (math): Find an equation for b in terms of a , s , and d and check it against your measured data.
- Open GeoGebra and create or move your sliders to your measured values of a and b . Using the values of a and b enter the canonical form of this ellipse.
- Use GeoGebra to calculate the *eccentricity* e of this ellipse using the formula $e = \text{Eccentricity}[c]$
- Use GeoGebra to calculate focus length f
 $f = ae$ for $a \geq b$ or $f = be$ for $a \leq b$ (2)
 Input: $f = \text{If}[a \geq b, a * e, b * e]$
- What is the relationship between f and d ? $d = \underline{\hspace{2cm}}$

Number a		a = 2
Number b		b = 3
Ellipse c	$x^2 / a^2 + y^2 / b^2 = 1$	$c: x^2 / 4 + y^2 / 9 = 1$
Number e	$\text{If}[a \geq b, \text{sqrt}((a^2 - b^2) / a^2), \text{sqrt}((b^2 - a^2) / b^2)]$	e = 0.75
Number f	$\text{If}[a \geq b, a * e, b * e]$	f = 2.24
Point F	$\text{If}[a \geq b, (f, 0), (0, f)]$	F = (0, 2.24)
Ellipse d	Ellipse with foci F, -F passing through (a, 0)	$d: x^2 / 4 + y^2 / 9 = 1$

Figure 5: Construction Protocol for Exercise 2

- Input coordinates for the focus F_1 into GeoGebra. Remember that F_1 must lie on the major (longer) axis so you need conditional input as we used for f .
- What is the relationship between F_1 and F_2 ? Input a formula for F_2 .
- Use input `Ellipse[focus1,focus2, point on ellipse]` to create a 2nd ellipse. (Use either (a,0) or (0,b) as your point). Do the ellipses coincide? They should.
- Challenge 2: Start with a rectangular piece of paper. Find the length of string and position of pushpins to create an ellipse *inscribed* in the rectangle?

D. GeoGebra Simulator of Drawing the Ellipse

Here we create a GeoGebra simulation of the drawing of the ellipse that we did with pushpins, string and paper. This is fun and connects the doing and the thinking.

In Figure 6, we show an image of our GeoGebra simulator.

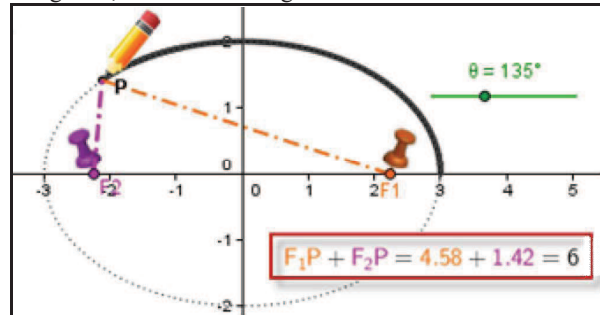


Figure 6: GeoGebra Simulation – Exercise 3

The challenging part here is to find a way to actually draw P in GeoGebra. It is too hard to find P using the “string”. The idea here is to see that the pencil goes 360° around $(0,0)$. So we look for a form of the ellipse which uses an angle θ as a parameter. In the canonical form we have $(x/a)^2 + (y/b)^2 = 1$ and we know that $\cos^2(\theta) + \sin^2(\theta) = 1$. So we let $x/a = \cos(\theta)$ and $y/b = \sin(\theta)$ and get a parametric form of the ellipse:

$$\begin{cases} x = a \cos(\theta) \\ y = b \sin(\theta) \end{cases} \text{ where } \theta=0 \text{ gives } (x,y)=(a,0) \quad (4)$$

Exercise 3: GeoGebra Simulation of Drawing an Ellipse

In Figure 7, we give the construction protocol for this construction. Images were found on Google.

Number a		a = 4
Number b		b = 3
Ellipse c	$x^2 / a^2 + y^2 / b^2 = 1$	$c: x^2 / 16 + y^2 / 9 = 1$
Number e	$\text{sqrt}((a^2 - b^2) / a^2)$	e = 0.66
Point F1	(a e, 0)	F1 = (2.65, 0)
Point F2	-F1	F2 = (-2.65, 0)
Angle θ		$\theta = 135^\circ$
Point P	(a cos(θ), b sin(θ))	P = (-2.83, 2.12)
Segment ap	Segment [F1, P]	ap = 5.87
Segment bp	Segment [F2, P]	bp = 2.13
Number string	ap + bp	string = 8

Figure 7: Construction Protocol for Exercise 3

III. HYPERBOLAS

Again, we start with the canonical form.

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \text{ where } a, b \in \mathbb{R}^+ \quad (4)$$

Exercise 4: GeoGebra and the Canonical Form of Hyperbola

- Start GeoGebra and open Algebra View
- Create sliders for a and b as positive real numbers.
- In the Input field, input: $x^2/a^2 - y^2/b^2 = 1$. The hyperbola will be drawn. As with the ellipse, by default in the Algebra view, the equation is written in 2nd order curve form. Right-click on the equation and (a) notice that GeoGebra recognizes that this is a hyperbola and (b) select the canonical form.
- Create points $A=(a,0)$, $B=(0,b)$, $C=(a,b)$ and $D=(-a,b)$. Create $O=(0,0)$ Draw a line through O and C and through O and D (Figure 8). Notice that these are the asymptotes. Find the equation of these lines using slope-intercept equations and compare with the equations in the Algebra view.
- Change the values of a and b and check that these relationships remaining true.
- Create points $E=(-a,-b)$ and $F=(a,-b)$ and a rectangle (polygon) with vertices C,D,E,F (Figure 9).
- Label the function and the asymptotes.
- As with ellipses, solve for y to get the explicit form of the hyperbola. Input these functions into GeoGebra (Figure 10).

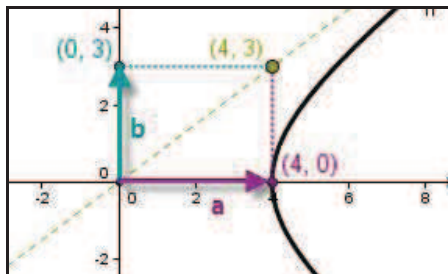


Figure 8: Understanding the Asymptote: $y=bx/a$

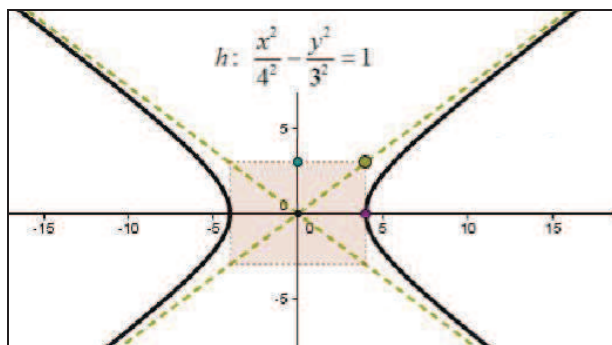


Figure 9: The Canonical Form with $a=4$, $b=3$

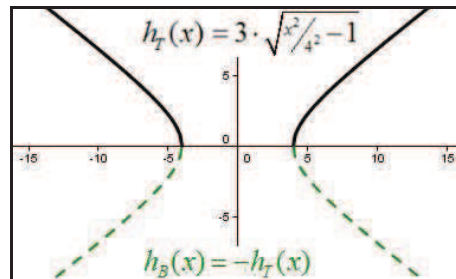


Figure 10: Explicit Form of the Hyperbola

We suggest that you also draw $y=1/x$ and $y=-1/x$ since these are indeed the hyperbolas used most frequently (see also Example 1 below.)

IV. HYPERBOLAS IN REAL LIFE

We give two examples where hyperbolas form in real life that we can use in our classroom.

Example 1: The first example is performed by taking 2 pieces of glass and sticking them flat together on the left side, while leaving a space of about 5mm between the plates on the right side. Place the plates in a pan filled with red beet juice and wait. Capillary action will pull the liquid up into the shape of a hyperbola (see Figure 11).

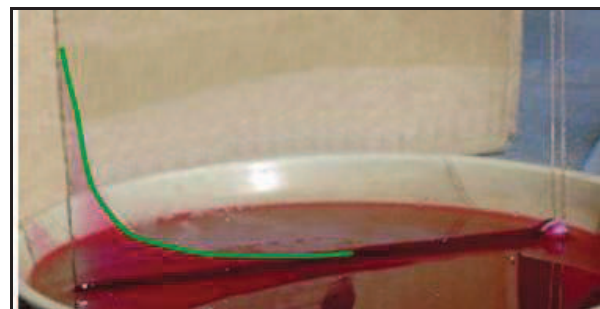


Figure 11: A Hyperbola Forms between Two Glass Plates [5]

Example 2: The second example from real life is a simulation of wave interference that the pupil can create using GeoGebra (Figure 11). Suppose 2 waves of equal amplitude and speed are generated from 2 distinct points at different times. For example: tap your left index finger into a pool of water and then your right index finger – same depth and speed of tap. At some point the waves will meet when they are in opposite phase, that is, the peak of one wave meets the trough of the other wave. At this point they will cancel each other out. Tracing this point of interference generates a hyperbola.

Using GeoGebra 4, we generate 2 circles at different points with a time difference of Δt . The radii of these circles increase at a constant rate like waves expanding at the same speed. For simplicity, we will assume that the circle(s) generated at the left point (green) represent the peak of those wave(s) and the circle(s) generated at the right point (red) represent the trough of those wave(s). In Figure 12, we give the construction protocol for this GeoGebra worksheet.

Number t	time
Number Δt	time delay between waves
Number a	$1/2$ distance of wave centers
Point A	$(a, 0)$
Point A'	$-A$
Circle c	Circle with center A and radius $t - \Delta t$
Circle d	Circle with center A' and radius t
Point B	Intersection point of d, c
Point C	Intersection point of d, c
Circle c1	Circle with center A and radius t
Circle d1	Circle with center A' and radius $t - \Delta t$
Point B1	Intersection point of c1, d1
Point C1	Intersection point of c1, d1

Figure 12: Construction Protocol for Wave Generator

In Figure 13-Figure 16, we show how a wave generated at the point $(-1,0)$ and another wave generated from the point $(1,0)$ but *later* (time delay $\Delta t + T/2$) spread, meet and then intersect. We trace their intersection points as the waves interfere and cancel. This curve of interference is a hyperbola.

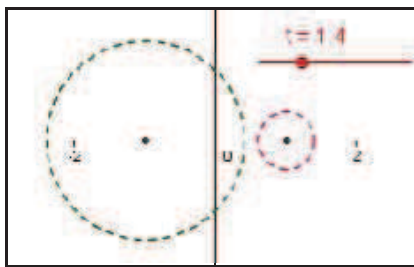


Figure 13: Two Waves are Generated with Time Delay

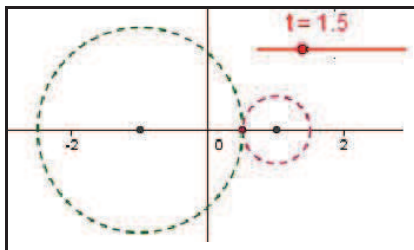


Figure 14: The Two Waves Meet - Interference Begins

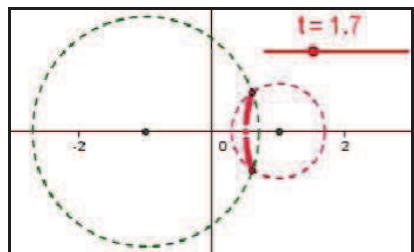


Figure 15: Tracing the Points of Intersection

In Figure 16 and Figure 17, we generate 2 waves at each point and show how the interference of the waves generated at different times result in 2-sided hyperbolas. In Figure 17, the

time delay between the generated waves has been cut in half with respect to Figure 16. Notice how this difference changes the hyperbola.

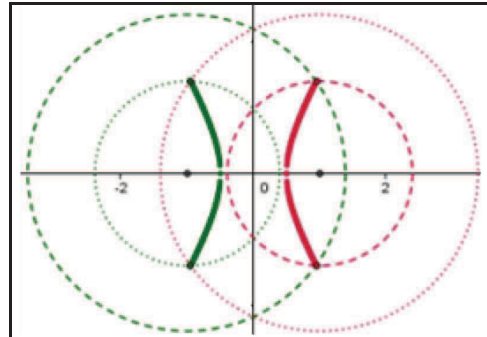


Figure 16: Interference of Waves Generated at $\Delta t=1[s]$

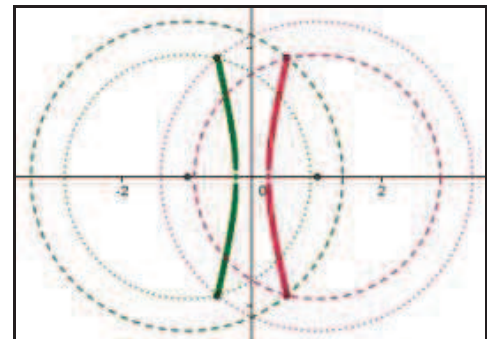


Figure 17: Interference of Waves Generated at $\Delta t=0.5[s]$

V. SUMMARY

In our discussion of ellipses, we used GeoGebra 4 to interactively explore the canonical, explicit and parametric forms of the ellipse, as well as consider how the foci and eccentricity of an ellipse are related to these forms. We did a hands-on project and then simulated this with GeoGebra. In our discussion of hyperbolas, we used GeoGebra 4 to interactively explore the canonical and explicit forms. We gave two examples of how hyperbolas are generated in real life and showed how to make a hands-on and GeoGebra simulation of these processes. In this way, we offer substance and form, interactivity and physical activity to the presentation of these important curves, which should increase interest and understanding.

REFERENCES

- [1] <http://www.geogebra.org/webstart/4.0/geogebra-40.jnlp>
 - [2] <http://en.wikipedia.org/wiki/Ellipse>
 - [3] <http://classroomk12.wikispaces.com/Draw+an+Ellipse>
 - [4] <http://en.wikipedia.org/wiki/Hyperbola>
 - [5] <http://paccadult.lbpsb.qc.ca/eng/extra/img/Hyperbolas.ppt>
 - [6] <http://classroomk12.wikispaces.com/Drinking+Hyperbola>
- All links reference on or after: 15.04.2011