COMPUTING COMPLEXITY OF A NEW TYPE OF THE DIAPHONY

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Abstract

In the present paper we introduce a new version of the diaphony, the so-called weighted $(W(b);\boldsymbol{\gamma})$- diaphony. It is shown that the $(W(b);\boldsymbol{\gamma})$- diaphony is a numerical measure for uniform distribution of sequences in $[0,1)^s$. The computing complexity $O(N^2)$ of the $(W(b);\boldsymbol{\gamma})$- diaphony of an arbitrary net, composed of N points in $[0,1)^s$ is obtained.

I. Introduction

We suppose that the notion of uniform distribution of sequences in the s-dimensional unit cube $[0,1)^s$ is well-known. For more information see Kuipers and Niederreiter [6].

The quantitative theory of the uniformly distribution of sequences studies the numerical measures for the irregularity of their distribution. The diaphony is an analytical measure for the irregularity of the distribution.

Following Chrestenson [2], for a non-negative integer k and a real $x\in[0,1)$, with the b-adic representations

$$k=\sum_{i=0}^{\nu}k_i b^i \quad \text{and} \quad x=\sum_{i=0}^{\infty}x_i b^{-i-1},$$

where for $i\geq 0$, ν_i, $k_i\in\{0,1,\ldots,b-1\}$, $k_\nu\neq 0$, the k-th **function of Walsh** ${}_b wal_k(x):[0,1)\to\boldsymbol{C}$ is defined as

$${}_b wal_k(x)=e^{\frac{2\pi I}{b}(k_0x_0+k_1x_1+\ldots+k_\nu x_\nu)}.$$

The set $W(b)=\{{}_b wal_k(x),\ :k=0,1,\ldots,\ x\in[0,1)\}$ is called the Walsh functional system in base b.

Let $\boldsymbol{N_0}$ be the set of non-negative integers, for an arbitrary vector $\boldsymbol{k}=(k_1,\ldots,k_s)\in\boldsymbol{N_0}^s$ and $\boldsymbol{x}=(x_1,\ldots,x_s)\in[0,1)^s$ we define the multivariate Walsh function in base b:

$${}_b wal_{\boldsymbol{k}}(\boldsymbol{x})=\prod_{j=1}^{s}{}_b wal_{k_j}(x_j).$$

Grozdanov and Stoilova in [5] introduced the b-adic diaphony, so for each integer $N\geq 1$ the b-adic diaphony $F_N(W_{(b)};\xi)$ of the first N elements of the sequence $\xi=(\boldsymbol{x}_i)_{i\geq 0}$ in $[0,1)^s$ is defined by

$$F_N(W(b);\xi)=\sqrt{\frac{1}{(b+1)^s-1}\sum_{\boldsymbol{k}\in N_0^s,\boldsymbol{k}\neq\boldsymbol{0}}\rho(\boldsymbol{k})\left|\frac{1}{N}\sum_{i=o}^{N-1}{}_b wal_{\boldsymbol{k}}(\boldsymbol{x}_i)\right|^2}$$

where for each vector $\boldsymbol{k}=(k_1,\ldots,k_s)\in N_0^s$, $\rho(\boldsymbol{k})=\prod_{\mu=1}^{s}\rho(k_\mu)$, and for an each integer $k\geq 0$

$$\rho(k)=\begin{cases}1;k=0\\ b^{-2g};\forall k,b^g\leq k<b^{g+1},g\in\boldsymbol{N}_0\end{cases} \qquad (1).$$

II. Number-theoretical motivations

Following Sloan and Wozniakowski [7], let $H_s(K)$ be a Hilbert space with a reproducing kernel $K:[0,1]^{2s}\to\boldsymbol{R}$. The multivariate integral

$$I_s(f)=\int_{[0,1]^s}f(\boldsymbol{x})d\boldsymbol{x},\ f\in H_s(K)$$

is approximated by quasi-Monte Carlo algorithm with equal quadrature weights

$$Q_s(f;P_N)=\frac{1}{N}\sum_{n=0}^{N-1}f(\boldsymbol{x}_n),$$

where $P_N=\{\boldsymbol{x_0},\boldsymbol{x_1}\ldots,\boldsymbol{x_{N-1}}\}$ is a deterministic sample point net composed of $N\geq 1$ points in $[0,1)^s$.

The worst-case error of the integration in the space $H_s(K)$ is defined as

$$e(H_s(K);P_N)=\sup_{f\in H_s(K),\|f\|_{H_s(K)}\leq 1}\left|I_s(f)-Q_s(f;P_N)\right|.$$

For an arbitrary vector $\boldsymbol{\sigma}\in[0,1)^s$ we define the so-called **(*s,b*)-**digitally **shifted net** $P_N((s,b);\boldsymbol{\sigma})$ by

$$P_N((s,b);\boldsymbol{\sigma})=\{\boldsymbol{x_0}\oplus_s^b\boldsymbol{\sigma},\boldsymbol{x_1}\oplus_s^b\boldsymbol{\sigma},\ldots,\boldsymbol{x}_{N-1}\oplus_s^b\boldsymbol{\sigma}\}.$$

Here for vectors $\boldsymbol{x},\boldsymbol{y}\in[0,1)^s$ the operation $\oplus_s^b$ means a b- adic "addition".

The **mean square worst-case error** $\hat{e}_{(s,b)-ds}(H_s(K);P_N)$ of the integration in the space $H_s(K)$ by using a random (s,b)-digitally shifted net P_N is defined by the equality

$$\hat{e}_{(s,b)-ds}(H_s(K);P_N)=\left(\int_{[0,1]^s}e^2(H_s(K);P_N((s,b);\boldsymbol{\sigma}))d\boldsymbol{\sigma}\right)^{\frac{1}{2}}.$$

The Hilbert spaces can be divided as Korobov spaces and Sobolev spaces. In general, the Korobov spaces are classes of functions with a given rate of convergence to zero of their Fourier coefficients, while the Sobolev spaces are classes of functions with smooth mixed partial derivatives.

The worst-case error and the mean square worst-case error of the integration in Hilbert spaces are related with the diaphony, as a characteristic of the quality of the distribution in $[0,1)^s$ of the point nets, which are used in the process of the integration.

In [4] the so-called weighted b- adic diaphony $F_N(\phi(b),\alpha;\boldsymbol{\gamma};\xi)$ as a numerical measure for uniform distribution of sequences has been introduced. This diaphony is based on the orthonormal functional system $\phi(b)$. Using the system $\phi(b)$ the weighted Korobov space $H_{\varphi(b),\alpha,\boldsymbol{\gamma}}$ is defined.

It is proved that the weighted b- adic diaphony is related with the worst-case error of the integration in Korobov spaces.

Now, we will discuss, what happens when we consider the mean square worst-case error of the integration in Sobolev spaces.

Balaz, Grozdanov, Dimitrievska Ristovska, Strauch and Stoilova [1] considered the mean square worst-case error of the integration in arbitrary reproducing kernel Hilbert space $H_s(K)$, which forms a weighted Sobolev space.

The obtained results are applied to the concrete Sobolev space $H_{Sob,s,\boldsymbol{\gamma},\beta_4}$ which is based on using the Bernoulli polynomials:

$$B_0(x)=1,\ B_1(x)=x-1/2,\ B_2(x)=x^2-x+1/6$$
$$\text{and } B_4(x)=x^4-2x^3+x^2-1/30.$$

After then, it is proved that there is a (t,m,s)-net P_{b^m} such that the mean square worst-case error of the integration in the space $H_{Sob,s,\boldsymbol{\gamma},\beta_4}$ by using a random (s,b)-digitally shifted net P_{b^m} has an order $O(b^{-m})$.

In order to improve the convergence rate of the mean square worst-case error, the tent transformation Φ in base b is introduced.

The mean square worst-case error of the integration in the space $H_s(K)$ by using first (s,b)-digitally shifted and second folded by the tent transformation net

$$P_N=\{\Phi(\boldsymbol{x}_0\oplus_s^b\boldsymbol{\sigma}),\Phi(\boldsymbol{x}_1\oplus_s^b\boldsymbol{\sigma}),\dots,\Phi(\boldsymbol{x}_{N-1}\oplus_s^b\boldsymbol{\sigma})\}$$

is considered. It is proved that, there is a (t,m,s)-net in base b P'_{b^m} such that the mean square worst-case error of the integration in the space $H_{Sob,s,\boldsymbol{\gamma},\beta_4}$ by using first (s,b)-digitally shifted and second folded by the tent transformation net P'_{b^m} has an order $O(b^{-2m})$.

Here, the essential fact is using the so-called (s,b)-digitally shifted and then folded by the tent transformation kernel $K_{(s,b)-ds;\Phi}$.

Especially, the k-th Fourier-Walsh coefficients of $(1,b)$-digitally shifted and then folded by the tent transformation kernel are given in a formulae with positive constants

$$c_1, c_2, c_3 \qquad (2)$$

defined in explicit form.

Our intention is to introduce a new weighted b-adic version of the diaphony, such that some well-known classes of sequences, for example the generalized Van der Corput sequence, to have an order, better than the corresponding order of the b-adic diaphony of these sequences. This will be realized if we use new coefficients, smaller than the coefficients (1).

We will use coefficients, which have the same order, as the order of the coefficients (2), to define new version of the diaphony.

For simplification of our work, the constant c_1,c_2,c_3 in (2), we will replace with the weight γ.

III. The weighted $(W(b);\boldsymbol{\gamma})$ - diaphony

In the next definition we will give the concept of the **weighted $(W(b);\boldsymbol{\gamma})$ diaphony.**

Definition 1.

Let $\boldsymbol{\gamma}=(\gamma_1,\dots,\ \gamma_s)$ where $\gamma_1\geq\gamma_2\geq\dots\geq\gamma_s>0$ be an arbitrary vector of positive weights. For each integer $N\geq 1$ the weighted $(W(b);\boldsymbol{\gamma};\xi)$ diaphony $F_N(W(b);\boldsymbol{\gamma};\xi)$ of the first N elements of the sequence $\xi=(\boldsymbol{x}_i)_{i\geq 0}$ in $[0,1)^s$ is defined by

$$F_N(W(b);\boldsymbol{\gamma};\xi) = \sqrt{\frac{1}{C(b,s,\boldsymbol{\gamma})}\sum_{\boldsymbol{k}\in N_0^s,\boldsymbol{k}\neq\boldsymbol{0}} R(b,\boldsymbol{\gamma},\boldsymbol{k})\left|\frac{1}{N}\sum_{n=o}^{N-1}{}_b wal_{\boldsymbol{k}}(\boldsymbol{x}_n)\right|^2},$$

where for each vector $\boldsymbol{k}=(k_1,\dots,k_s)\in\boldsymbol{N}_0^s$, $R(b,\boldsymbol{\gamma},\boldsymbol{k})=\Pi_{j=1}^{s}\rho(b,\gamma_j,k_j)$, and for a real $\boldsymbol{\gamma>0}$ and an arbitrary integer $k\geq 0$ the coefficient $\rho(b,\gamma,k)$ is defined by:

$$\rho(b,\gamma;k)=\begin{cases}1;\ if\ k=0\\ \gamma b^{-4a};\ if\ k=k_{a-1}b^{a-1},k_{a-1}\in\{1,..,b-1\},a\geq 1\\ \gamma b^{-2(a+j)}+\gamma b^{-4a};\ if\ k=k_{a-1}b^{a-1}+k_{j-1}b^{j-1}+\\ k_{j-2}b^{j-2}+\dots+k_0;k_{a-1},k_{j-1}\in\{1,..,b-1\},a\geq 2,j\geq 1\end{cases}$$

where

$$C(b,s,\boldsymbol{\gamma})=\prod_{h=1}^{s}[1+\frac{\gamma_h(b+2)}{b(b+1)(b^2+b+1)}]-1.$$

Theorem 1 will show the fact that the weighted $(W(b);\boldsymbol{\gamma};\xi)$-diaphony is a numerical measure for uniform distribution of sequences.

Theorem 1:

The sequence $\xi=(\boldsymbol{x}_i)_{i\geq 0}$ of points in $[0,1)^s$ is uniformly distributed if and only if the equality

$$\lim_{N\to\infty}F_N(W(b),\boldsymbol{\gamma},\xi)=0,$$

holds for each vector of weights $\boldsymbol{\gamma}$.

For arbitrary reals $x,y\in[0,1)$ let $\oplus_1^b$ denotes the operation "digit-by-digit" addition modulo b, and for arbitrary vectors $\boldsymbol{x},\boldsymbol{y}\in[0,1)^s$ let $\oplus_s^b$ denotes coordinate-by-coordinate $\oplus_1^b$ addition. For arbitrary reals $x,y\in[0,1)$ we will introduce the next two conditions:

$(C1)$ $x\oplus_1^b y$ is not a b- adic rational;

$(C2)$ x and y are b- adic rationals.

The b- adic logarithm of $x\in[0,1]$ will be denoted by $\log_b x$.

If x has the b- adic representation

$$x=\frac{x_g}{b^g}+\frac{x_{g+1}}{b^{g+1}}+\dots,$$

where for $i\geq g$ $x_i\in\{0,1,\dots,b\text{-}1\}$ and $x_g\neq 0$ then the integer part of $\log_b x$ is defined as $-g$, so $\lfloor\log_b x\rfloor=-g$.

Theorem 2:

Let us for an arbitrary real $\gamma>0$ construct the function

$$\omega(b,\gamma,x)=1+\frac{\gamma(b+2)}{b(b^2+b+1)}-\frac{\gamma(b+1)(b^2+1)}{b(b^2+b+1)}b^{3\lfloor\log_b x\rfloor}+$$
$$\gamma\sum_{\alpha=2}^{\infty}b^{-2\alpha}\sum_{\beta=1}^{b-1}\sum_{\tau=1}^{\alpha-1}b^{-2\tau}\sum_{\mu=1}^{b-1}\sum_{l=\beta b^{\alpha-1}+\mu b^{\tau-1}}^{\beta b^{\alpha-1}+(\mu+1)b^{\tau-1}-1}{}_b wal_l(x),\ x\in[0,1).$$

For an arbitrary vector of weights $\boldsymbol{\gamma}=(\gamma_1,\dots,\ \gamma_s)$ where $\gamma_1\geq\gamma_2\geq\dots\geq\gamma_s>0$ we define the function

$$\Omega(b,\boldsymbol{\gamma},\boldsymbol{x}) = -1 + \prod_{h=1}^{s} \omega(b,\gamma_h,x_h),\ \boldsymbol{x}=(x_1,\ldots,x_s)\in[0,1)^s.$$

For an arbitrary integer $N\geq1$ let $\xi_N=\{\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_{N-1}\}$ be an arbitrary net composed of N points in $[0,1)^s$, such that for $0\leq n\leq N-1$ the coordinates of the points $\boldsymbol{x}_n$ satisfy the condition ($C1$) or ($C2$), in particular, the coordinates of all points are b-adic rationals.

Then, the weighted $(W(b);\boldsymbol{\gamma};\xi)$- diaphony $F(W(b);\boldsymbol{\gamma};\xi_N)$ of the net ξ_N satisfies the equality

$$F^2(W(b);\boldsymbol{\gamma};\xi_N) =$$

$$\frac{1}{C(b,s,\boldsymbol{\gamma})}\frac{1}{N^2}\sum_{n=0}^{N-1}\sum_{m=0}^{N-1}\Omega(b,\gamma,\boldsymbol{x}_n \ominus_s^b \boldsymbol{x}_m).$$

Theorem 2 gives the computing complexity $O(N^2)$ of the weighted $(W(b);\boldsymbol{\gamma})$-diaphony of an arbitrary net, composed of N points in $[0,1)^s$.

IV. Proofs of the Main Results

The proof of Theorem 1 is similar to the proof of Theorem 1 of [5].

Proof of Theorem 2:

We can prove the following equalities:

$$\int_0^1 b^{3\lfloor \log_b x\rfloor} dx = \frac{1}{(b+1)(b^2+1)}$$

and for each integer k, such that $b^{a-1}\leq k<b^a$

$$\int_0^1 b^{3\lfloor \log_b x\rfloor}\, \overline{{}_b wal_k(x)}\, dx = -\frac{\mathrm{b}(\mathrm{b}^2+\mathrm{b}+1)}{(b+1)(b^2+1)} b^{-4a}.$$

By using of the above two equalities, we can prove that for each integer k, the Fourier-Walsh coefficients

$$\hat{w}_{W(b)}(b,\gamma,k) = \int_0^1 w(b,\gamma,x)\, \overline{{}_b wal_k(x)}\, dx$$

of the function $w(b,\gamma,x)$ satisfies the equality

$$\hat{w}_{W(b)}(b,\gamma,k) = \rho(b,\gamma,k).$$

After then, for each vector $\boldsymbol{k}\in\boldsymbol{N}_0^s$ the Fourier-Walsh coefficients of the function $\Omega(b,\boldsymbol{\gamma},\boldsymbol{x})$ satisfies the equality

$$\hat{\Omega}_{W(b)}(b,\boldsymbol{\gamma},\boldsymbol{k}) = \begin{cases} 0, \boldsymbol{k}=\boldsymbol{0} \\ R(b,\boldsymbol{\gamma},\boldsymbol{k}), \boldsymbol{k}\neq\boldsymbol{0} \end{cases}$$

Here the coefficients $\rho(b,\gamma,k)$ and $R(b,\boldsymbol{\gamma},\boldsymbol{k})$ are defined in Definition 1.

So, the function $\Omega(b,\boldsymbol{\gamma},\boldsymbol{x})$ has the Fourier-Walsh expansion

$$\Omega(b,\boldsymbol{\gamma},\boldsymbol{x}) = \sum_{\boldsymbol{k}\in N_0^s, \boldsymbol{k}\neq\boldsymbol{0}} R(b,\boldsymbol{\gamma},\boldsymbol{k})\, {}_b wal_{\boldsymbol{k}}(\boldsymbol{x}),\ \forall \boldsymbol{x}\in[0,1)^s \qquad (3).$$

Let now $\xi_N=\{\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_{N-1}\}$ be an arbitrary net composed of N points in $[0,1)^s$.

By using the equality (3) we have the presentation

$$\frac{1}{C(b,s,\boldsymbol{\gamma})}\frac{1}{N^2}\sum_{n=0}^{N-1}\sum_{m=0}^{N-1}\Omega(b,\gamma,\boldsymbol{x}_n \ominus_s^b \boldsymbol{x}_m) =$$

$$= \frac{1}{C(b,s,\boldsymbol{\gamma})}\frac{1}{N^2}\sum_{n=0}^{N-1}\sum_{m=0}^{N-1}\sum_{\boldsymbol{k}\in\boldsymbol{N}_0^s,\boldsymbol{k}\neq\boldsymbol{0}} R(b,\boldsymbol{\gamma},\boldsymbol{k})\, {}_b wal_{\boldsymbol{k}}(\boldsymbol{x}_n \ominus_s^b \boldsymbol{x}_m)$$

$$= \frac{1}{C(b,s,\boldsymbol{\gamma})}\sum_{\boldsymbol{k}\in N_0^s,\boldsymbol{k}\neq0} R(b,\boldsymbol{\gamma},\boldsymbol{k}) \frac{1}{N}\sum_{n=0}^{N-1} {}_b wal_{\boldsymbol{k}}(\boldsymbol{x}_n) \frac{1}{N}\sum_{n=0}^{N-1} \overline{{}_b wal_{\boldsymbol{k}}(\boldsymbol{x}_m)}$$

$$= F^2(W(b);\boldsymbol{\gamma};\xi_N)$$

V. References

[1] V. Balaz, V. Grozdanov, V. Dimitrievska Ristovska, O. Strauch, S. Stoilova, "On the mean square worst-case error of the integration in weighted Sobolev spaces," *to submit*

[2] H. E. Chrestenson " A class of generalized Walsh functions", *Pacific J. Math,* 1955, vol.5, pp. 17-31.

[3] H. Faure, Discrepances de suite associees a un system de numeration (en dimension un), *Bull. Soc. Math. France*, 1981, vol. 109,pp. 143-182.

[4] V.S. Grozdanov, "The weighted *b*- adic diaphony", *Journal of Complexity*, 2006, vol. 22, pp. 490-513.

[5] V. Grozdanov, S. Stoilova, "The *b*- adic diaphony", *Rendiconti di Matematica,* Serie VII, 2002, vol.22, pp. 203-221.

[6] L. Kuipers, H. Niederreiter "*Uniform Distribution of Sequences*", Wiley, New York, 1974.

[7] I. H. Sloan, H. Wozniakowski, "When are quasi-Monte Carlo algorithms efficient for high dimensional integrals?", *J. Complexity*, 1998, vol.14, pp.1-33.